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NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

FINAL REPORT (EM-97)

Basic Research in Electromagnetic Theory

CONTRACT No. AF19(122)-42

SEPTEMBER 1956

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under

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Subject:

Basic Research in Electromagnetic Theory

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Project Director

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Abstract

This final report on Contract No. AF 19(122)-42 summarizes basic mathematical investigations conducted by the Division of Electromagnetic Research of the Institute of Mathematical Sciences during the period January 1, 1949 to September 1, 1956. The research covers the subjects of diffraction theory, tropospheric and ionospheric propagation, microwave optics, mathematical methodology, and some specialized results on antenna theory and wave guides.

One of the objectives of this report is to make clear the patterns of research which we have pursued and which are somewhat obscure in the chronological sequence in which the reports appeared. The chief results of the various separate pieces of work are discussed within the framework of the larger goals and directions which this group has pursued.

Table of Contents

1. Introduction

During the period from Jan. 1, 1949 to Sept. 1, 1956 the Division of Electromagnetic Research of the Institute of Mathematical Sciences of New York University has pursued research in electromagnetic theory under Contract No. AF 19(122)-42. In accordance with the contract the objective has been to undertake mathematical investigations in the general field of electromagnetic theory with emphasis on such subjects as diffraction, tropospheric and ionospheric propagation, microwave optics, mathematical methodology and a number of other more specialized subjects which were specified in various modifications of the contract.

The nature of the work was rather unusual and perhaps warrants some discussion. It is well known that electromagnetic theory is securely founded on sound mathematical principles, namely, Maxwell's equations. It is also true that electromagnetic principles are basic in the design of equipment used for radar, communication including television, navigation, fire control, missile detection and control, weather forecasting, upper atmosphere research, and a variety of other military and civilian needs. Unfortunately the solution of electromagnetic problems involves considerable mathematical difficulty so that despite the fact that the subject is now about 75 years old, the number of problems which have been solved exactly and the number of methods readily applicable to routine ways are severely limited. It has therefore been deemed desirable that progress be made in several directions: The development of mathematical theory which would be potentially applicable to theoretical problems in electromagnetics, the creation and formulation of methods for the solution of specific types of electromagnetic problems, the application of advanced mathematical techniques to difficult electromagnetic problems, and the solution of specific problems, techniques for which are known to professional mathematicians.

Our group has endeavored to fulfill these objectives. The nature of this work, however, has led not to the investigation and completion of one specific task but to a variety of results, some largely mathematical and others physically quite concrete; moreover, these results cover different subjects within electromagnetic theory. Partly for this reason and partly because the chronological order of the reports issued during the life of the contract fails to reveal the main directions and goals, this final report on the work will attempt to organize the contents of the reports so as to make clear the goals pursued and the rationale and continuity of our efforts. In view of the fact that the

ultimate objective of our work is to aid practicing physicists and research and development engineers, this report has classified the separate results in accordance with the physical applications to which they are relevant. Our research has, of course, been intimately related to that of numerous other investigations in this country and elsewhere. The relationships and relevant bibliographies are in the individual reports.

Prior to the undertaking of this work and concurrently with it our group did other work in electromagnetics and in related mathematical fields. In particular, some of the purely mathematical work begun under this contract was continued under a separate contract with the Office of Scientific Research, A.R.D.C. An account of this earlier and more recent work, along with a digest of past and current work under this contract as of May 1955, will be found in the publication of the Symposium on Electromagnetic Theory held at the University of Michigan in June 1955 and published in the Transactions of the Professional Group on Antennas and Propagation, I.R.E., July 1956, pp. 243-63. This publication was submitted as Report EM-93 under the contract.

Though our primary effort under this contract has been to strengthen the science of electromagnetic theory we have been constantly aware of the advisability of attracting talented young people to this field and to train them for future professional work. A number of young people were supported in part during their graduate work by funds allotted to us under the contract and received doctoral degrees at New York University. Some of these people received further training through post-doctoral research. The names of these people and their current affiliations are:

Alfred Leitner, Asso.Prof. of Physics, Michigan State University
William Sollfrey, Midway Laboratories, University of Chicago
Jerry Shmoys, Asst.Prof. Elec. Eng., Polytechnic Institute of Brooklyn
Jerome Lurye, Staff member, Technical Research Group, New York City
Arnold Russek, Asst.Prof. of Physics, Univ. of Connecticut
Milton Rose, Office of Naval Research, Washington, D.C.
Herbert B. Keller, Division of Computing Services, Inst.Math.Sci., N.Y.U.
Irvin Kay, Res. Associate, Div. of Electromagnetic Res., New York Univ.
Jack Bazer, Res. Associate, Div. of Electromagnetic Res., New York Univ.
Albert Blank, Asst.Prof. of Math., Univ. of Tennessee
George Kear, General Electric, California
Lester Kraus, Consolidated Vultie Aircraft Corp., San Diego, Calif.
Mortimer Weitz, Devenco, Inc., New York City

During the past few years four young men partly supported by this contract have been doing graduate work at New York University and preparing for careers in electromagnetic theory. Their names are James Radlow, Bertram Levy, Robert Lewis, and Bernard Seckler. Several young people who were doing graduate work while being supported under the contract did not complete their doctoral work, at least under our auspices. Among these are Mrs. Joy Bruno Russek now instructor in mathematics at the University of Buffalo, Mr. Chester Marcinkowski now with the Microwave Research Institute of the Polytechnic Institute of Brooklyn, and Stanley Preiser now with Nuclear Development Corp. of America, White Plains, N.Y. Several young men joined us after receiving their Ph.D's elsewhere and improved their research strength while undertaking work under the contract. In this group are Dr. Victor Twersky now with Sylvania Electric Products Inc., Mountain View, California and Dr. Nicholas Chako now lecturer in mathematics at Queens College, Flushing, N.Y.

Before proceeding to discuss the technical work done under the contract we should like to make one comment concerning the state of electromagnetic theory. Subjects such as tropospheric and ionospheric propagation, diffraction theory, antennas, and wave guides have been investigated since about 1900. However the research was carried on by isolated individuals who, under whatever influences determine the career of a scientist, happened to choose these fields. This work has been sporadic, disconnected, and fragmentary. On the other hand when emergencies such as World War II and international tensions created great needs for electromagnetic theory, large scale efforts were made; but these efforts have on the whole been confined to the solution of problems arising in applications already in view. Hence the work has been limited, on the whole, to obtaining the most practical solutions which existing theory permits. Though such efforts may always be needed under special circumstances and often yield returns far beyond what is sought or expected, they are not the substitute for long range, continuing, reasonably sized, basic investigations. It is a truism in the history of science that free exploration of ideas, even those which are not at all promising for applications produces the greatest advances. Moreover science generally has expanded so much that each branch has pursued its own investigations independently of and almost in ignorance of what other branches are doing. Yet the theoretical ideas developed in one branch may be extremely valuable in another. Hence it is important that those who seek to advance electromagnetic theory be enabled to inform themselves to some extent of the progress in acoustics, hydrodynamics,

quantum mechanics, and optics, and, of course, mathematics. The group of mathematicians and physicists working under this contract has been extremely fortunate in obtaining far-sighted, wise, and sympathetic support from a number of scientists of the Air Force Cambridge Research Center. We should like to thank Nathaniel C. Gerson, Francis J. Zucker, Nelson Logan and Philip Newman. Their participation in the formulation, direction, and conduct of the work has enabled us to function as members of a university staff while attempting to serve the interests and needs of the laboratories at the Center. We are happy to acknowledge also extremely pleasurable and cordial relationships with these men.

2. Mathematical Methodology

A. Field Solutions in the Quasi-Optical Range. The rapid development of ultra-high frequency and microwave techniques has given rise to hundreds of new electromagnetic problems. The frequencies involved are, of course, close to the optical range and hence electromagnetic theorists have utilized geometrical optics methods, which were extensively explored during the two hundred years before the birth of electromagnetic theory. The solutions obtained thereby are necessarily approximate and often unsatisfactory. Hence, from the very outset of our work we have sought a systematic attack on electromagnetic problems which would take advantage of the existing optical methods and yet improve on them in the direction of wave theory; that is, we have sought to develop the transition from optics to electromagnetics. The link between these two domains, which has proved to be both theoretically and practically significant, is the notion, due to R. K. Luneberg, of asymptotic solution of electromagnetic problems (Luneberg, EM-14,15). In this approach one seeks not exact solutions of electromagnetic problems due to time harmonic sources but asymptotic series solutions in which the basic variable in the series is the wavelength λ or the reciprocal of the circular frequency ω of a monochromatic source.

The first major step in attempting an asymptotic series solutions of an electromagnetic problem is to determine the form of the asymptotic series which is valid for the particular type of problem, e.g., reflection, refraction, diffraction by edges, and diffraction by smooth bodies. The second step is to learn how to calculate these series. To accomplish the first objective we have utilized the concept of the pulse solution of Maxwell's equations. The pulse field corresponding to an electromagnetic field due to an aggregate of charges in a finite domain or charges at infinity having a harmonic time behavior $e^{-i\omega t}$ is the field due to the very same charges but having a Heaviside unit function time behavior, that is, 0 for $t < 0$ and 1 for $t > 0$. By means of the relationship between the pulse solution and the steady state time harmonic solution we first derived the proper form of the asymptotic series in those spatial regions where reflection and refraction only (in the geometrical optics sense) takes place. In the cases of directly transmitted and reflected fields and refracted fields the asymptotic series which give the spatial behavior of E and H (the time behavior is $e^{-i\omega t}$), respectively, consist essentially of terms in the zeroth and positive integral powers of $1/\omega$, each multiplied by a phase factor. In the case of fields diffracted

by an aperture or lens we showed that the series may contain positive fractional powers of $1/\omega$. It is of mathematical importance at least that the coefficients of the successive terms of the asymptotic series are the discontinuities of the pulse solution and of its successive time derivatives respectively.

The asymptotic form of fields diffracted by obstacles in homogeneous media such as cylinders, spheres, and other convex bodies in free space has posed a more difficult problem. At the present time our results limit the forms which obtain in such fields (Friedlander, Keller, EM-67). The chief characteristic of these forms is an exponential decay factor and fractional exponents in the series for $1/\omega$. The exponential factor (apart from a phase factor) is $e^{-k^{\alpha}\lambda}$ where $0 < \alpha \leq \frac{1}{2}$ and $\lambda(x,y,z) = $ const. are certain determinable equi-amplitude surfaces. The fractional powers of $1/\omega$ which appear in the series depend upon the value of α which is needed for a particular problem and are determinable in terms of α. If, for example, $\alpha = 1/2$ then the powers of $1/\omega$ are of the form $(1/\omega)^{n/2}$. If $\alpha = 1/3$ the powers of $1/\omega$ are of the form $(1/\omega)^{n/3}$. For other values of α, the powers are not simply stated.

The problem of determining the coefficients of these several types of asymptotic series is approached by studying their variation along curves orthogonal (in isotropic media) to surfaces of equal phase. These surfaces are the wavefronts of geometrical optics and satisfy the eiconal equation of optics, namely, $(\nabla\phi)^2 = \varepsilon\mu$, where $\phi(x,y,z) = $ const. is the family of fronts and ε and μ which may also depend upon x,y,z are the dielectric constant and permeability of the medium. The family of orthogonal curves is then the family of rays of geometrical optics.

We have sought therefore to extend the theory of wavefronts and rays. In an early report (Keller and Keller, EM-13) we extended the theory of ordinary geometrical optics (or acoustics) to find the intensity of the reflected and transmitted fields due to a point source in the presence of an arbitrary interface between two media. A particular consequence of the solution is the general lens and mirror law and the equations for the caustic surfaces. In a further extension (Keller and Preiser, EM-20) we have considered an incident field having an arbitrary wavefront which impinges on an arbitrary interface between two media, and have computed the amplitudes of the reflected and transmitted fields. The result includes the special cases of plane and spherical incident wavefronts which were

treated previously.

Since it has been part of our program to extend the theory and application of asymptotic solutions to diffraction problems and since the theory of wavefronts and rays had been confined to reflection and refraction we undertook a more serious extension of ray theory to diffracted fields (Keller, EM-92 and EM-94). This extension shows what rays must be introduced in the shadow region created by smooth bodies such as cylinders, spheres, and other convex bodies, the rays introduced by diffraction at an edge, and the rays produced by diffraction at a corner of an obstacle. In all of the above cases if one takes a point P on the incident ray and a point Q on a diffracted ray then the optical path from P to Q renders Fermat's integral stationary under the appropriate boundary condition that the path must include a point or curve on the diffracting surface.

Having the families of rays we are now prepared to trace the variation of the individual coefficients of the asymptotic series along these rays. The essential result here is that these coefficients each satisfy a first order linear ordinary differential equation in which the independent variable is arc length or some other suitable parameter along the ray. Moreover, the entire set of ordinary differential equations forms a recursive system. The non-homogeneous term of any one differential equation is known in terms of the solutions of the preceding equations. A rather complete derivation of these ordinary differential equations for both scalar and vector problems and for the case of incident, reflected and refracted fields was obtained (Kline, EM-24,48). For diffracted fields the derivation of these ordinary differential equations is formal. To obtain the unique solutions of these differential equations initial values are needed. These can be obtained for many classes of problems but there are still difficulties in, so to speak, starting the asymptotic series along rays in the diffracted region.

The merit of the entire scheme of obtaining asymptotic series solutions by the above method lies in the fact that one replaces the problem of solving an exact boundary value problem in Maxwell's equations by the problem of obtaining an asymptotic series solution, the latter requiring only the solution of a system of first order ordinary linear differential equations. (One does not need to know the pulse solution at all.) Moreover the first term of this series is the geometrical optics solution where this exists and the higher terms improve on

geometrical optics.

There is, however, another way in which the theory can be applied. The theory states under rather general conditions how, if one knows the behavior of the pulse solution in the neighborhood of its singularities, one can write down at once the asymptotic series solution of the corresponding time harmonic problem (Luneberg, EM-14). If therefore one wishes to find such an asymptotic solution and if one can find the behavior of the pulse solution in the neighborhood of its singularities, then the desired answer can be written down at once. Such a procedure may be applied to problems wherein the pulse solution or at least its singularities are more readily obtainable. This is the case as we shall point out in other connections.

There is another application. In the theory of diffraction by lenses or by an aperture the answer to the time harmonic problem can be expressed (on the basis of a Kirchhoff assumption) in the form

$$\iint_D g(p,q)\, e^{i\omega f(p,q)} dp dq,$$

where p,q are direction cosines of any rays entering the image space, f and g are known functions, and D is the domain of all p,q for the given lens or aperture. One can show (Luneberg, EM-15) that the pulse solution of this same problem is

$$P(t) = \iint_{f(p,q) \le ct} g(p,q) dp dq$$

By analyzing the behavior of P(t) in the neighborhood of its singularities one obtains the information needed to write down the asymptotic expansion of the double integral in inverse integral and fractional powers of $1/\omega$. When g(p,q) is analytic the singularities of P(t) depend upon the behavior of the contour lines f(p,q) = const. in the domain D. These singularities are known in the literature as critical points of the diffraction integrals above.

We have been able to proceed from theory to concrete applications of the asymptotic theory. Reference will be found below under work on diffraction theory to the fields reflected by infinite smooth bodies (Keller et al EM-81), to the determination of fields at foci and caustics of antennas (Kay and Keller,

EM-55), to the fields diffracted by finite smooth bodies such as cylinders (Keller, EM-94) and to the field diffracted by an aperture (Keller, EM-92, EM-96).

B. Wiener-Hopf Methods. During the past war a number of difficult waveguide problems were solved by recasting the natural boundary value problem involving partial differential equations into the Wiener-Hopf integral equation by means of Green's theorem and by solving this integral equation. The essential feature of the method of solving the integral equation is the use of certain techniques of complex function theory. However the scope of the entire method and its natural domain of applicability seemed to be obscure.

One of our reports (Karp, EM-25) analyzes the inner mechanism of this method, characterizes its legitimate range of application, and demonstrates its connection with the method of separation of variables. Boundary value problems are generally attacked so that the physical boundary is one entire coordinate surface. In this case the usual method of separation of variable can be employed at least in principle. However, there are problems in which the boundary does not consist of one entire coordinate surface but, for example, the condition $u = 0$ may have to be satisfied on a half-line and $u_n = 0$ on the other half. We refer to such a problem as a two-part boundary value problem. Moreover, the choice of coordinate system may determine whether a given problem is a one-part, two-part, or even three part boundary value problem. Normally the Wiener-Hopf technique is successful with two-part boundary value problems. However, this report shows that the separation of variables (or eigenfunction) technique when supplemented by function-theoretic techniques (such as are involved in the Wiener-Hopf method) not only solves such problems but is entirely parallel to the Green's function, Wiener-Hopf technique. For example, diffraction by a staggered array of an infinite number of semi-infinite parallel plates, solved in the past by the Wiener-Hopf technique, is solved in this paper by the function-theoretic extension of the method of separation of variables. The paper also shows how the integral equation to which one is led by the use of the Green's function may be solved by the use of transforms other than the Fourier transform used in the Wiener-Hopf method. Thus the infinite ribbon may be treated in polar coordinates as a two part problem and by use of the Mellin transform. (It can of course be treated as a one-part problem in elliptic coordinates.) A number of old and new problems are used as illustrations in this paper.

The understanding of the role of the Wiener-Hopf technique obtained through the above analysis made clear that new problems could be solved. These applications (Karp EM-35; Bazer and Karp, EM-46 and EM-66; Williams, EM-77) will be discussed below.

C. Equivalence Principles. Among methodological studies we have devoted some time to what may broadly be called equivalence principles. By this term we mean any principle which allows us to obtain the solution of one electromagnetic problem from the solution of another through some simple relationship between the two types of problems. The method of images, Babinet's principle and the reciprocity theorem are well known examples of equivalence principles. Incidentally, in connection with the last mentioned principle, though it is used freely, precise formulations of the theorem for various situations are often lacking. A proof of the applicability of this theorem to propagation in anisotropic media was obtained (Lurye, EM-31). A new type of equivalence principle (Karp and Williams, EM-83) which involves the application of Schwartz's reflection principle to achieve analytic continuation across a straight boundary shows how new types of problems are solvable from known solutions. The functions involved in the electromagnetic problems are solutions of the reduced wave equation (Helmholtz equation) and these are usually complex functions $u(x,y)$ of real arguments x and y. In the present equivalence principle, $u(x,y)$ is regarded as an analytic function of complex variables x and y. The function theoretic properties of this $u(x,y)$ have been investigated by mathematicians and such properties as analyticity and analytic continuation by reflection and other principles of continuation are known to the mathematical literature. These properties are used in the paper under discussion. The reflection principle says that the unique analytic continuation of a function $u(x,y)$ across a plane on which $u = 0$ is given by an odd reflection, e.g., $u(x,y) = -u(x,-y)$, and the unique continuation across a plane on which $u_n = 0$ is given by an even reflection, e.g., $u(x,y) = u(x,-y)$. Applications to wave guide and diffraction problems will be given below.

D. Variational Methods. Some attention has been given to variational solutions of electromagnetic problems. At the outset of our work when knowledge of the application of such methods in electromagnetic theory was limited we

presented an expository paper on the method proper (Sollfrey, EM-11). This paper shows how to obtain the back-scattering cross section of arbitrary obstacles by variational means. The problem of computing back-scattering leads to a problem in differential equations. This formulation is recast into an integral equation by means of Green's theorem. A general theorem then relates the solution of the integral equation to the problem of rendering stationary a certain ratio of integrals - a typical variational formulation. By the use of suitable trial functions the variational procedure can be made to yield approximations to the value of the back-scattering cross section of the obstacle. Some simple first order computations for sound and electromagnetic waves incident on an obstacle are made in this paper is to show how the variational procedure can be employed; full advantage of the procedure for calculations was not taken in this paper.

More recently a critique of the variational method (Jones, EM-78) has shown that the variational method of dealing with the integral equations of scattering problems is equivalent to solving the integral equation directly by Galerkin's method and using the standard formula for the amplitude of the scattered wave. The second method also satisfies the reciprocity theorem. It is therefore suggested the the reciprocity theorem be used as the basis of approximation without the introduction of variational formulas. The error involved in using an approximate solution is discussed and it is shown that only a special set of approximations can lead to accuracy at low frequencies. Some ways in which bounds for the error may be obtained in special problems are also given.

E. Differential Equations. Since the problem of electromagnetic wave propagation in the stratified ionosphere involves the solution of a system of four ordinary differential equations some attention has been paid to obtaining explicit as well as more useful approximate solutions of such systems. Insofar as improved approximations are concerned we have treated the following problems (Keller and Keller, EM-33).

The system of ordinary differential equations is represented as

$$\frac{dU(z)}{dz} = A(z)\, U(z), \qquad U(z_0) = U_0,$$

in which $U(z)$ is an n-rowed column vector and $A(z)$ is an $n \times n$ square matrix whose elements are functions of real or complex z. (When the elements of A are

constants the general solution is, of course, the usual linear combination of n exponential functions.)

This research starts with the Peano-Baker matrizant representation of the solution, the matrizant of a matrix $A(z)$ being defined as

$$\underset{z_o}{\overset{z}{\Omega}} \left\{A(\chi)\right\} \equiv I + \int_{z_o}^{z} A(\chi_1)\, d\chi_1 + \int_{z_o}^{z} A(\chi_1) \int_{z_o}^{\chi_1} A(\chi_2) d\chi_2\, d\chi_1 + \ldots$$

The solution which can be expressed directly in terms of the matrizant converges too slowly and so is transformed by matrix operations to one which converges more rapidly. By further transformation it is possible to write any component $u_i(z)$ of $U(z)$ as

$$u_i(z) = \sum_{j=1}^{n} \left[\frac{p_{ij}(z)\, e^{\int_{z_o}^{z} g_{jj}(\chi)\, d\chi}}{\det P(z)} \right] e^{\int_{z_o}^{z} \lambda_j(\chi)\, d\chi}\, C_j(z_o,z) \det P(z_o).$$

The full explanation of this expression is too lengthy to be presented; however, the main feature of this form is that each component of the solution is a sum of exponential functions with variable exponents and variable coefficients. C_j is one component of a column matrix and is itself a sum of matrizants and hence a sum of a series of multiple integrals.

If the matrix $A(z)$ contains k, where $k = 2\pi/\lambda$, in such a form that $A(z,k) = kA(z)$ then integration by parts applied to the above solution yields an asymptotic series in positive integral powers of $1/k$. Hence both convergent and asymptotic series representations of $u_i(z)$ are obtained. The result is a natural generalization of the W.K.B. approximation and shows that this approximation is the first term in both the asymptotic expansion and in the true series solution. An asymptotic expansion of the solution is obtained for a special parametric dependence; it is similar to Birkhoff's result for an n-th order linear ordinary differential equations. The equation:

$$\frac{d^2y}{dx^2} - 2a(x) \frac{dy}{dx} + b(x)y = 0$$

is solved as an application of the method.

Physical application of this theory will be discussed below under ionospheric

propagation. The theory of systems of ordinary differential equations was continued under another contract because of the highly mathematical nature of the work.

Under the subject of differential equations we made some start on the theory of ordinary differential equations utilized rather uncritically in papers on tropospheric propagation. Information concerning the eigenvalues of the relevant second order ordinary differential equation is needed most. Hence a study was made (Phillips, EM-42) of the operator $L[y] = -y'' + q(x)y = 0$, where y is in $L_2(0,\infty)$ and satisfies the boundary conditions $y(0) = 0$ and $\lim\limits_{x \to \infty} (\alpha y + \beta y') = 0$, where α/β need not be real. For $q(x)$ real, two situations arise depending on $q(x)$. In the limit point case y is always independent of α/β, L is self-adjoint and has a real spectrum. In the limit circle case L has a pure point spectrum which lies in the upper half plane if $\mathrm{Im}[\alpha/\beta] > 0$ and in the lower half plane if $\mathrm{Im}[\alpha/\beta] < 0$. An elementary divisor theory for operators with completely continuous resolvents is considered. For $q(x)$ complex, with $|\mathrm{Im}[q(x)]|$ bounded, the operator L is treated by a perturbation procedure starting with the operator L_0 in which $q_0(x) = \mathrm{Re}[q(x)]$. Representation theorems for some classes of operators are also considered.

In behalf of work done on the subject of ionospheric propagation and other problems to be discussed below we have pursued and extended research on differential equations concerned with the following problem. Given the differential equation

$$u''(k,x) + [k^2 - V(x)]\, u(k,x) = 0$$

defined over the interval $-\infty < x < \infty$ and subject to the conditions that a solution exist of the form

$$e^{ikx} + b(k)\, e^{-ikx} \qquad \text{for } x \to -\infty,$$

to determine $V(x)$. This 'inverse' problem has been solved by Gelfond and Levitan for the semi-infinite interval $0 \leq x < \infty$. We have solved this problem for the infinite interval subject to the condition that $V(x) \equiv 0$ for x less than any given but fixed number a (Kay, EM-74). Given $b(k)$, which physically is a reflection coefficient as a function of the wave number, as an analytic function of k (it is analytic under proper assumptions on $V(x)$) it is possible to determine $V(x)$. The function $b(k)$ may also be a function of the angle of incidence α at

a fixed frequency in which case k involves α. Further, if $b(k)$ may be approximated by a rational function, then $V(x)$ can be obtained in closed form.

This theory can be applied to tropospheric and ionospheric propagation, wave guides with variable cross-section, transmission lines with variable inductance and capacitance, propagation through a slab with dielectric variable in the direction perpendicular to the faces (e.g., the radome problem), and to the synthesis problem of circuit theory, for example, the design of filter networks. Some of these applications are discussed below.

Two expository papers in the field of differential equations were written to make accessible techniques used by theoreticians in electromagnetics and other branches of physics. The first report (Friedman, EM-47) discusses the theory of the Dirac delta function and singular functions which play an important part in classical applied mathematics and in modern physics. Although the results are in most cases well-known, it has been thought worthwhile to present a fairly rigorous mathematical treatment of these topics in order to give the physicists confidence in the use of the delta function. The treatment is based on the theory of distributions which was developed by Laurent Schwartz.

The second report (Friedman, EM-60) continues the investigation of Research Report No. EM-47 on the applications of recently developed mathematical techniques. The aim here is to present the theory of Green's functions for ordinary differential equations and for systems of ordinary differential equations. This theory can be used to obtain the spectral representation of ordinary differential equations and to solve partial differential equations.

F. Integral Equations. Though most of our work on integral equations has been segregated under another contract because of the mathematical nature of the work, we refer here to the study of Galerkin's method of obtaining an approximate solution (Jones, EM-78) mentioned in connection with variational methods and to another method of treating integral equations arising in low frequency diffraction theory (Jones, EM-87). The standard integral equation of the first kind which arises in scattering problems seeking a scattered field behaving like a radiating wave is converted into an integral equation of the second kind by splitting the solution into a non-radiating field which approaches the static limit as the wavelength becomes infinitely long and the difference between the desired field and this non-radiating field. The new integral equation has a particularly simple kernel so that, although it is difficult to solve exactly,

it can be solved approximately with ease. A general theory is developed of the corresponding integral equation which would arise in <u>any</u> scattering problem. It is shown that upper bounds can be obtained for the scattering amplitude and the scattering coefficient.

G. <u>Miscellaneous Mathematical Studies</u>. Expansions or 'addition theorems' for the spherical wave functions

$j_n(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$, $h_n^{(1)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$, and $h_n^{(2)}(kR)\, P_n^{|m|}(\cos\theta)\, e^{im\phi}$,

with reference to the origin O, have been obtained in terms of spherical wave functions with reference to the origin O', where O' has the coordinates (r_o, θ_o, ϕ_o) with respect to O (Friedman-Russek, EM-44). Using the above-mentioned addition theorems, we have obtained an expansion for $\frac{e^{ikr_{12}}}{r_{12}}$, a wave which is spherically symmetric about the source Q, where Q is referred to the origin O, in terms of products of spherical waves with respect to origin O and spherical waves with respect to O'.

The further study of addition theorems for other types of special functions such as the parabolic cylinder and paraboloidal functions was continued on another contract devoted to mathematical work.

Starting from a general integral theorem for spheroidal functions, we have obtained (Chako, EM-73) several integral relations involving products of spheroidal functions by choosing the kernel in the integral to be of the form $e^{-i\mu'\phi}Lu$, where u is a solution of the time-harmonic wave equation in spheroidal coordinates and L is an operator commuting with the Laplacian. Operators possessing this property are certain linear combinations of the linear and the angular momentum operators and powers and products of these combinations. Several new integral relations between associated Legendre functions have been obtained as limiting cases of the integral relations involving products of spheroidal functions.

3. <u>Diffraction Theory</u>

We embrace under this heading the various studies of the reflection and diffraction of radio waves by smooth surfaces, apertures, gratings, and antenna problems.

We call attention first to a survey and critique of recent developments (Bouwkamp, EM-50) which embraces much recent literature on the subject. A number of papers in the theory of diffraction of electromagnetic waves, particularly those dealing with apertures in plane conducting screens, are reviewed. The subjects treated include modifications of Kirchhoff's theory, the theory of small apertures, Babinet's principle for plane obstacles, variational principles, and singularities at sharp edges.

A. Scattering from Smooth Bodies. The obviously practical application to radar cross-sections has motivated a great deal of work on the scattering of radio waves by smooth bodies. We include here bodies with edges because the emphasis is on the reflected and scattered field rather than on any limited study of edge effects which are treated in section B.

Several studies were devoted to the fields diffracted by smooth bodies when the source is a pulse. The pulse can be a point or line source, located in a finite place which at $t = 0$ suddenly rises to full strength and propagates toward the diffracting obstacle, or it can be a plane front (source at infinity) which is of unit strength behind the front and zero in front as it moves toward the diffracting body. The interest in pulse solutions for us has stemmed from the fact that pulse solutions are helpful in understanding the nature of the asymptotic series solutions required for various types of problems and from the fact that one can obtain steady state solutions from pulse solutions by a Fourier transform with respect to time and sometimes the resulting steady state field so obtained is calculable.

Several diffraction problems involving pulse sources were devoted to the conducting wedge, a shape which has both mathematical and practical interest.

The diffraction and reflection of plane electromagnetic pulses by perfectly conducting wedges and corners is treated by a direct method (Keller and Blank, EM-21). The results also apply to acoustic pulses with rigid or free walls. First the propagation of the discontinuity at the front of the pulse is investigated. This enables us to transform the initial - boundary value problem to a characteristic - boundary value problem in xyt space. The geometry is such that the problem can then be solved by the method of conical flow commonly used in supersonic aerodynamics. An explicit solution in closed form involving only elementary functions is obtained.

The diffraction of a plane pulse by a conducting wedge or cone is

obtained by first finding the solution due to an incident time harmonic plane wave and then integrating over the propagation constant of this wave (Sollfrey, EM-45). The integrals which appear may all be evaluated. For the wedge the solution agrees with that previously obtained by Keller and Blank. For the cone, the solution is given as an infinite series in the diffraction region. Outside this region the series solution may be summed to give the expected discontinuous solution.

The preceding work on diffraction of pulses by a conducting wedge was generalized to cover an arbitrary pulse wave incident in a direction normal to the generators of a perfectly conducting cylindrical wedge (Kay, EM-43). Either the electric vector or the magnetic vector of the incident wave is polarized parallel to the generators of the cylinder. The field is desired for all time.

This problem can be reduced to that of finding a solution of the wave equation, $\Delta u - \frac{1}{c^2} u_{tt} = 0$, subject to boundary conditions of the form $u = 0$ or $\frac{\partial u}{\partial n} = 0$ on the obstacle and subject to a certain inhomogeneous boundary condition on a characteristic surface of the wave equation. The inhomogeneous boundary condition which occurs on the characteristic surface is determined by Luneberg's discontinuity condition. This condition is implied by the requirement that the discontinuous solution and its discontinuous first time derivative be limits of continuous solutions and their continuous first time derivatives respectively. These conditions, along with the shape of the obstacle, also serve to determine the particular characteristic surface which occurs. In fact, at the edge there appears just the characteristic cone of the wave equation, the cone whose half vertex angle has a slope equal to the propagation speed c.

The diffraction of plane pulses by a parabolic cylinder and by a paraboloid of revolution was investigated for both the boundary conditions $u = 0$ and $\frac{\partial u}{\partial n} = 0$ (Kay, EM-53). In all cases the pulse is assumed to be incident on the vertex of the diffracting obstacle in a direction parallel to its axis. The problems are treated by first separating variables in the time dependent wave equation and then solving the resulting integral equations by means of a Laplace transform. In each case a Volterra integral equation for the solution is obtained, which can be solved by the usual iteration process. It is shown that when the parabolic cylinder degenerates into a half plane, the solution reduces to that obtained for the latter problem by other methods.

Another paper (Friedlander, EM-64) deals with the diffraction of a

sound pulse due to a line source by a parallel, rigid, fixed circular cylinder. This problem can also be interpreted in electromagnetic terms. The diffracted field is resolved into terms representing diffracted pulses which have encircled the cylinder a number of times in either sense, and these in turn are expressed as a series of 'propagation modes' with the same pulse fronts. Approximations valid near the front of a mode are then obtained. The corresponding results for an incident plane pulse are derived by a limiting process. It is found that a diffracted pulse has an essential singularity of a definite type at its fronts, similar to the intial stages of diffusion, and that the pressure rises very gradually at first. But after time intervals of the order of one-third to one-half of the radius of the cylinder divided by the velocity of sound, there seems to be no appreciable diffraction effect.

Of course in practice time harmonic sources are most important. However except for the simplest of geometries exact solutions even in terms of series of special functions are not available and even in the case of simple geometries, the half-plane, cylinder, and sphere, the same series solution is generally not useful for all frequencies or for near and far fields. Moreover, even where useful, often the known solutions require a great deal of calculation. We have therefore sought to exploit the method of asymptotic series solution which should be useful in the high frequency range since the series are asymptotic to the correct time harmonic field for large frequency and in fact yield the geometric optics field, when this exists, for infinite frequency.

We have treated first a number of diffraction problems in which the diffracted field is entirely within the space illuminated by the source, that is, the diffracted field is covered by reflected and refracted rays of classical geometrical optics (Keller, Lewis, and Seckler, EM-81). These problems include diffraction of a plane wave by a parabolic cylinder, a paraboloid of revolution, a cylinder and a sphere; diffraction of a spherical wave by a paraboloid of revolution, a hyperboloid of revolution, and a plane interface; diffraction of a cylindrical wave by a parabolic cylinder, a hyperbolic cylinder and a plane interface, etc. The boundary conditions considered are the vanishing of the function or of its normal derivative and the impedance boundary condition. Formulas are obtained for reflection of any wave from any two dimensional surface, and certain formulas are deduced for three dimensional problems.

The problem of diffraction by smooth bodies of finite cross-section wherein there is a true diffracted field is of course much more difficult even,

as noted above, for simple geometries. Nevertheless considerable progress has been made in obtaining the asymptotic field diffracted by convex bodies of finite cross-section (Keller, EM-94).

The leading term in the asymptotic expansion for large $k = 2\pi/\lambda$, of the fields reflected and diffracted by any convex cylinder are constructed. The cross section of the cylinder is assumed to be a smooth curve which may be either closed or open and extending to infinity. The method employed is an extension of geometrical optics in two respects. First, diffracted rays are introduced. Secondly, fields are associated with the rays in a simple way. The results are applicable when the wavelength is small compared to the cylinder dimensions.

Some work has been done on the more traditional approaches, that is, convergent series solutions, of obtaining the field diffracted by perfectly conducting obstacles. One study considers a new approach to the problem of scattering of high frequency plane waves by a sphere in the forward direction (Kear, EM-86).

An expression for the scattered wave as an expansion in terms of radial eigenfunctions is obtained. The total wave may be expressed in terms of radial eigenfunctions directly but the incident wave cannot. Instead, the incident wave is first expressed by a contour integral. Then a change of variable is introduced and the resulting integral is approximated by the Euler-Maclaurin sum rule. This results in a series for the incident wave in terms of radial functions plus an integral and correction terms. When this is subtracted from the total wave a finite series in terms of radial functions is obtained, and the integral and correction terms in the expression for the incident wave are easily evaluated.

The problem of diffraction of electromagnetic waves by a cylindrically tipped perfectly conducting wedge under time harmonic plane wave or line source excitation has also been treated (Karp, EM-52). Mathematically, this reduced to the problem of finding a solution of the two-dimensional time-reduced wave equation which either vanishes or has a vanishing normal derivative on the surface of the tipped wedge depending upon whether the electric vector or the magnetic vector is normal to the plane of incidence. Series representations of the solutions are obtained in all cases in terms of the eigenfunctions of the simple wedge problem (where the radius of the cylindrical tip is zero). The far field expressions for the case of the cylindrically tipped wedge under plane wave excitation are obtained for both polarizations. Graphs of the net scattered amplitude are given for the special case of normal incidence on a half-plane with a cylindrical

tip of radius $a = k^{-1}$, k being the wavenumber.

Diffraction by a T-shaped obstacle with the vertical part infinite in extent has been shown by means of the reflection principle to be equivalent to the solution of the two independent problems, the half-plane and the infinite strip (Karp and Williams, EM-83). An explicit solution is obtained for a special case of a wave normally incident on the top surface of the T.

A special application of diffraction by a perfectly conducting wedge is the corner reflector which is intended to reflect radiation primarily in the incident direction. Using geometrical optics methods we have extended the theory of this device (Keller, EM-36). All configurations of two or three mirrors meeting at a point and having the property that any ray incident on them is reflected back parallel to its original direction, are determined. The results include the well-known cases of two and three mutually orthogonal mirrors, as well as other cases, apparently new. It is also shown that there are no configurations of more than three mirrors meeting at a point, which have this property. The results generalize those of Synge who showed that the mutually orthogonal case was the only three mirror configuration having the parallel reflection property, provided every ray strikes each mirror exactly once. The new configurations occur if the number of reflections is not restricted.

Diffraction by dielectric obstacles is of interest for such purposes as radomes, sealing off wave guide slots and flush mounted antennas. The problem of the dielectric wedge in particular has excited interest because while so much has been learned about the perfectly conducting wedge and, to some extent, the imperfectly conducting wedge, the dielectric wedge has defied exact solution. An approximate solution has been obtained (Karp and Sollfrey, EM-33) in the case where the dielectric constant differs but slightly from that of its surroundings. The fields are expanded in powers of the difference in dielectric constant and only first powers are retained. The approximate fields are then represented as a superposition of plane waves of arbitrary complex direction of propagation, and the resulting integrals evaluated by stationary phase to obtain the asymptotic form of the scattered fields. The reflected and transmitted plane waves agree with those obtained by geometrical optics, and the angular pattern of the diffracted cylindrical waves is found.

The radome problem calls for the design of dielectrics which should be perfectly transparent. Through work done on ionospheric propagation to be described below, we obtained theoretical results on the construction of

dielectrics which are completely reflectionless (Kay and Moses, EM-91). A complete solution is given for the problem of constructing a plane stratified dielectric medium having the property that at a fixed frequency and polarization a plane wave incident at any angle will be transmitted without reflection by the medium. The dielectric medium is infinite in both directions perpendicular to the planes of stratification. It is proved that the index of refraction of all such reflectionless media must be of the form

$$\frac{\sum A_n e^{ax}}{\sum B_n e^{ax}} .$$

Though our own interest in reflection and transmission by anisotropic layers was motivated originally by ionospheric propagation problems, there are current projected applications of such layers to the design of absorbing media. We might call attention here to our work on scattering by a continuously varying anistropic medium (Russek, EM-38) further described below under ionospheric propagation.

Further work on diffraction by smooth bodies is described below in connection with propagation around a cylindrical or spherical Earth.

B. Diffraction by Apertures and Discs. Diffraction by apertures and discs, though in many respects the same type of problem as diffraction by smooth bodies, may be regarded as another class of problems partly because it poses special mathematical difficulties such as edge singularities and partly because the applications for which the theory is intended are different. Slots in wave guides, used for coupling of guides or for antenna arrays at ultra-high frequency, edge effects in paraboloidal antennas, and the lens problem are all basically aperture diffraction problems. Because it has thus far proved more amenable to exact and approximate solutions, the circular aperture has received the largest share of attention both in our group and outside. The general survey article (Bouwkamp, EM-50) contains a critique and comparison of various approaches to the circular aperture and disc problems.

Though exact solutions for the scalar and vector circular aperture problem are now known interest in approximate solutions continues partly because calculations by the exact solution are extremely lengthy and at present out of the question for large apertures, so that even verification of the geometrical optics limit is analytically difficult, and partly because approximation methods

useful in many other electromagnetic problems can be tested on the aperture problem.

In an appendix to EM-50, Marcuvitz gives a systematic formulation of the vector problem of diffraction by an aperture which makes the solution depend upon a vector integro-differential equation for the transverse (parallel to screen) electric field in the aperture. The aperture field is obtained by solving first an inhomogeneous transverse vector partial differential equation and then an inhomogeneous vector integral equation of the first kind. These equations are solved for the circular aperture by a perturbation on the static limit to obtain the zeroth order and first order terms in k.

We have also considered (Levine, EM-84) a plane wave normally incident on the circular aperture in soft or hard screens ($\Psi = 0$ or $\partial_n \Psi = 0$ on the screen) and have sought the explicit lowest order correction, in reciprocal powers of ka, to the geometrical optics transmission cross section. The method is the variational one applied to an integral equation formulation of these problems. Thus, for the soft screen, an integral equation describing the distribution of $\partial_n \Psi$ on the shadow face affords a convenient basis. By appropriate modification of the Green's function occurring in the integral equation, the high frequency behavior of $\partial_n \Psi$ can be obtained, and utilized for the calculation of the transmission cross-section. The first approximation to $\partial_n \Psi$ is that pertaining to a local infinite straight edge at each direction tangent to the aperture rim, as proposed by Braunbek on physical grounds. In the next approximation, a supplement to the Braunbek distribution is again associated with a straight edge configuration, where however, the excitation arises from a (secondary) plane wave at grazing incidence. The grazing wave describes an interaction between points at (roughly) opposite ends of an aperture diameter. It is noteworthy that even a first correction to the geometrical cross-section obtained in this way proves reliable at moderate frequencies, a feature well known in other instances of asymptotic expansions. This circumstance contrasts markedly with the behavior at low frequencies, where successive terms in a power series expansion do not significantly extend the qualitative or quantitative aspects of the cross-section.

The correction is sensitive to the screen boundary condition, and in this paper details are given, for normal incidence only and for the acoustically 'soft' screen, where the wave function vanishes. There are two ways of proceeding; namely, 1) to use equations which characterize the normal derivative of the wave function at the screen, and 2) to start with the wave function itself in the

aperture. As a consequence of approximations employed, there is some disagreement in the results. The screen formulation turns out to possess advantages, and predicts a cross-section

$$\sigma \cong \pi a^2 \left(1 - \frac{2}{\sqrt{\pi}} \; \frac{\sin(2ka - \frac{\pi}{4})}{(ka)^{3/2}}\right) \qquad ka \gg 1,$$

where a is the aperture radius. It is shown that the straight-edge screen distribution due to a grazing plane wave, and not the primary wave, plays the decisive role for the correction in the cross-section. A comparison with other approximation techniques is given, and emphasis is placed on the role of boundary conditions at the aperture rim.

Levine and Schwinger have shown that the field in the circle aperture, which determine the diffracted field, can be determined by the solution of an infinite system of linear equations for certain unknowns D_m ($m = 0, 1, 2, \ldots$). The D_m are power series in $2\pi a/\lambda$, where a is the radius of the aperture and λ is the wavelength. We have considered this infinite system of equations (Magnus, EM-32) and have shown that, by solving the first ℓ equations for the first ℓ unknowns, the exact values of the first ℓ coefficients of the power series for $D_0, D_2, \ldots, D_{\ell-1}$ are obtained. Explicit recurrence formulas are given. It is shown that the solution is unique and convergent for sufficiently small values of $2\pi a/\lambda$. The mathematical nature of the system of linear equations is investigated by showing that it is connected with a problem of moments for a finite interval. It is proved that in a limiting case the transmission coefficient can still be computed although the asymptotic form of the system has no solution at all for the D_m.

The type of infinite system of linear equations arising in the Levine-Schwinger approach to the circular aperture problem has been studied further (Magnus, EM-80). The coefficients of this system depend on a parameter α. It is proved that the solution of the system can be found to any degree of approximation by solving the first N equations for the first N unknowns, for N sufficiently large, if α is either real or purely imaginary.

The theory required to calculate the field diffracted by an aperture and the transmission cross-section is so extensive even in approximate methods that we have regarded it worth while to see if the asymptotic series method

already described would be simpler. At the same time we have achieved an extension of this method to a new class of problems (Keller, EM-92). The diffraction of a wave by an aperture of <u>any</u> shape in a thin screen is treated by this new method, which may be called 'the geometrical theory of diffraction,' because it is an extension of geometrical optics which accounts for diffraction. In this method new rays - called diffracted rays - are introduced. They are produced when an incident ray hits the edge of the aperture, and they satisfy the 'law of diffraction.' A field is associated with each ray in a quantitative way, by means of the optical principles of phase variation and energy conservation. In addition 'diffraction coefficients' are introduced to relate the field on a diffracted ray to that on the corresponding incident ray.

By this method a simple formula is obtained for the field diffracted by any aperture. By means of this formula the field in the aperture or the far field diffraction pattern can be found. In addition the transmission cross section of the aperture can be determined. Explicit formulas and numerical results are given for slits and circular apertures. The accuracy of the results increases as the wavelength decreases, but they are even useful for wavelengths as large as the aperture dimensions.

A comparison of results obtained by the asymptotic series method just described with those obtained by the Kirchhoff and modified Kirchhoff methods and the use of the Braunbek correction, was made (Keller, EM-96). In each case a double integral over the aperture is evaluated asymptotically, and contributions from interior stationary points, edge stationary points, and corners of the edge are obtained. The contributions from points of each type, according to all the theories examined, are exactly of the form recently deduced by J. B. Keller using his geometrical theory of diffraction. The edge stationary points and the corners correspond respectively to rays singly diffracted from edges and from corners. For edge diffracted rays the Braunbek diffraction coefficients alone coincide with those given by the geometrical theory of diffraction, but the Braunbek method does not apply to corner diffraction. None of these methods takes account of multiple diffraction as does the geometrical theory of diffraction.

The analysis of the Wiener-Hopf integral equation method led as noted to the realization that some new problems can be solved by the method. One of these is the problem of fluid flow through a conical pipe which proves to have an application to diffraction theory (Bazer and Karp, EM-66). The primary objective of this report is to obtain an exact solution of the following

potential problem: Find an axially symmetric solution of the potential equation having a vanishing normal derivative on the surface of a semi-infinite conical pipe with a circular aperture at one end and having a prescribed behavior at infinity. The solution may immediately be interpreted as the velocity potential of a steady-state irrotational flow of a non-viscous, incompressible fluid through a rigid conical pipe with a circular aperture. Using this interpretation, the solution is employed to derive, for suitable excitation, an approximate expression for the far field of the corresponding boundary-value problem involving the diffraction of sound.

The method is to construct integral representations of the solution using a variant of the Wiener-Hopf procedure. These representations lead to eigenfunction expansions of the potential from which exact expressions for the hydrodynamical conductivity of the opening are obtained. The behavior of the velocity of the fluid near the circular edge of the conical pipe is determined and is employed to prove the uniqueness of the solution. Using the Rayleigh static method, which requires the knowledge of the conductivity of the opening, we obtain the approximate far field 'outside' the pipe resulting from excitation 'inside' the pipe.

The disc and aperture problem are related by Babinet's principle (see EM-50). Hence some results for the aperture problem can be obtained by treating the complementary problem of the disc, and the latter is of interest in itself. Under our investigation of integral equations we found a method of splitting the desired radiated field into a standing wave (non-radiating) solution plus a residual field (Jones, EM-87). This method was applied by the author to the problem of finding the field scattered by a rigid circular disc when an axially symmetric sound wave, either a point source on the axis or a plane wave, impinges on it. The scattering coefficient is obtained approximately, with good agreement with the exact solution when the product of the wave number and radius of the disc is less than or equal to 3. Estimates of the error in calculating the pressure distribution on the disc, the scattered amplitude, and the scattering coefficient are readily made.

In another investigation of the disc problem (Leitner, EM-12) the exact theory is extended to compute and discuss values for the diffracted field both at the disk and at large distances from it. The Kirchhoff solution - an approximation designed for the case of very short wavelength - is compared to the exact solution and it is found to be more powerful than might heretofore

have been supposed. The reasons for this are discussed in some detail.

The rectangular aperture widely used in waveguide applications is more difficult to treat and much theory has therefore been directed to the somewhat simpler problem of diffraction by a slit in a conducting plane. The slit is infinite in one direction and, of course, finite in the other. One of our investigations treats the slit as two half-planes appropriately separated (Karp and Russek, EM-75). A rigorous solution of this problem in terms of Mathieu functions has been given by Morse and Rubinstein and by Skavlem, and approximate solutions have been given by Sommerfeld, Bouwkamp, and Groschwitz and Hönl. If d is the <u>half-slit</u> width, λ is the wavelength, and $K = 2\pi/\lambda$, then we can say that the rigorous solution is useful for $Kd < 10$, while the approximate solution mentioned above is useful only for much smaller kd. On the other hand Schwartzschild has demonstrated a series of successive approximations which converges for all kd, but is especially useful in the limit $kd \to \infty$. Schwartzschild evaluated the leading term, which is useful only in the optical limit and which represents the half planes as scattering independently. The present work bridges the gap in the range of kd covered by existing solutions and obtains the form of the correction term in the expansion of the solution for large kd. It proved possible to show <u>a priori</u> that the correction term could be represented as a superposition of the fields scattered by each half-plane separately, when each half plane is under the influence of a line source of a suitable complex constant amplitude.

Approximate expressions for the near and far fields, taking into account the interaction between the edges, are derived in terms of the well-known solutions for the field produced when an isolated conducting half-plane is excited by a) a plane wave, and b) a line source. Results of numerical calculation are given for the case of a plane wave normally incident on the slit and compared with the exact solution obtained by Mathieu functions, and with the results obtained when interaction is neglected. A comparison of transmission coefficients is also given. It is found that the new approximate solution agrees well with the exact solution, and provides a significant correction to the non-interaction solution. The accuracy increases with the slit width, so that the result is useful in the range where interaction cannot well be neglected but where the exact solution converges so slowly that computation is impracticable. A brief discussion of the case of line-source excitation is included.

The problem of diffraction of electromagnetic waves by a slit in a

perfect conductor at the interface between two different dielectric media has been solved by expanding the fields in each medium in terms of appropriate Mathieu functions (Meixner, EM-68). By matching these expansions across the slit an infinite set of linear equations is obtained for the coefficients in the expansion of the diffracted fields. The solution is put in such a form that numerical results can be readily obtained.

In electromagnetic diffraction problems involving edge effects the question of what singularities can occur and what condition on them is needed to define a unique and physically realistic solution has loomed large. Edge singularities had been considered for conductors in a uniform surrounding medium. Two of our reports (Karp, EM-71 and Meixner, EM-72) consider the singularities which can occur at edges of conducting bodies when the medium changes in the angular neighborhood of the edge. The first of these papers contains an investigation of the effect of a local discontinuity of dielectric constant upon the singularity in field strength in the neighborhood of a geometrical singularity of a conductor. The work is confined to the electrostatic case. We first consider the case of a wedge-shaped conductor in the presence of a plane surface of discontinuity passing through the vertex of the wedge. For this problem the dependence of the singularity on the various parameters is studied in detail, and in addition the Green's function is derived. It is found that a) when the wedge is symmetric with respect to the interface the singularity is unaffected by the presence of a discontinuity in the medium. It is also noted that b) in the symmetric case the Green's function can be expressed, by the method of images, in terms of the Green's function for a conducting wedge in free space. The result b) is then extended to the case of an arbitrary symmetric distribution of conducting material. The corresponding extension of the result a) is then deduced.

In the second of these papers the behavior of an electromagnetic field in the neighborhood of the common edge of angular dielectric or conducting regions is determined from the condition that the energy density must be integrable over any finite domain (the so-called edge condition). Two cases are treated in detail, namely, (1) a region consisting of a conducting wedge and two different dielectric wedges with a common edge, and (2) a region consisting of two different dielectric wedges with a common edge. It is also shown that near such edges electrostatic and magnetostatic fields will exhibit the same behavior as the electromagnetic field.

C. Diffraction by Combinations of Obstacles. Many electromagnetic problems involve diffraction by a combination of obstacles. The best known example is a grating which consists of identical elements generally understood to be equally spaced. Even if the problem of diffraction by a single element is entirely solvable, the field diffracted by the entire grating still presents considerable difficulty because each element receives not only the incident field but fields diffracted by every other element.

The interest in gratings stems originally from optical applications. However there are radio frequency problems in which gratings occur. Moreover there is considerable value even for optical applications in treating the grating as an electromagnetic problem because the theory based on Maxwell's equations is more accurate and may shed light, for example, by exhibiting the dependence of the diffracted field on frequency, where geometrical optics methods do not. There are also periodic structures such as arrays, series of slots, and ridged surfaces which can be regarded as gratings each of whose elements has its own scattering pattern. Moreover parallel plate wave guide problems in which the boundary condition on the walls calls for zero normal derivative of the magnetic field can be regarded as grating problems in which the normal derivative of the field halfway between each pair of elements of the grating and perpendicular to the grating elements is likewise zero (see, for example, EM-83). A brief review of earlier work on gratings by individuals outside our group will be found in Shmoys, EM-18.

Some of our efforts on the problem of combinations of obstacles have assumed that the field scattered by a single element is known or can be obtained experimentally and our object has been to determine the field diffracted by the combination. In the first of these papers (Karp, EM-85) diffraction by an infinite grating of congruent arbitrarily shaped conducting cylinders is shown to be related to single scattering by an isolated typical grating element, even when all interactions among elements are taken into account. It is assumed that the largest diameter of an element and the wave length are small compared to the spacing. Fourier amplitudes of the diffracted field are given in terms of differential scattering amplitudes of the single cylinder; the latter are either analytically calculable (e.g., for the circular cylinder) or obtainable by experiment.

In this paper the theory considers first two obstacles and expresses the

field diffracted by the combination as the sum of the incident field, the field diffracted by each obstacle when only the incident field impinges, and the field diffracted by each obstacle and arising from the field diffracted by the other. This third summand is the correction term to single scattering theory. The coefficients of the correction term are quotients with denominators of the form $1 + O\left(\frac{1}{\sqrt{kd}}\right)$ where d is the distance between the centers of the two obstacles. If the denominators are simplified by ignoring the higher order terms we obtain a general formula for first order multiple scattering which may be adequate in some problems. However, as noted below, there are cases where the higher order terms should not be neglected.

The above method holds even if the size of each body is comparable to or larger than the spacing provided each body subtends a small angle when viewed from any other body of the combination of obstacles, for example, two half-planes forming a slit. In this case the respective subtended angles are 0 even though the bodies are infinite in size. Insofar as the slit problem is concerned, strictly each body is excited by a line source rather than a plane wave. However there is a little known reciprocity theorem which states that the far field in the direction θ of a line source is expressible in terms of the near field of a plane wave incident from the direction θ and evaluated at the line source. When the line source is itself remote we can replace the near field of the plane wave by the far field of the plane wave, thus obtaining a further simplification because the coupling effect still involves only the response of cylinders to plane waves (see Karp and Russek, EM-75).

If the higher order terms in the denominator of the correction term are neglected, then the result is good only on the assumption that there is no spectrum in the plane of the grating. This so-called resonance case (Karp and Radlow, EM-90) results under normal incidence when the wavelength equals the spacing. In this case the procedure of simplifying the denominator will not work because the sum of the terms to be neglected become infinite. Thus the first order multiple scattering result is inaccurate. When the full denominator in the correction term is retained the resulting expression is indeterminate. We have however evaluated the limit as resonance is approached. The results involve certain second and third order determinants formed from the array of single scattered amplitudes. The vanishing or non-vanishing of the determinants is critical for the calculation. Detailed consideration of the result shows that, in contrast to non-resonance case, interaction can never be neglected near resonance.

The intensity in each spectrum for wavelengths near resonance can be markedly different from that for other wavelengths, and our formulas exhibit this feature as well as a dependence on the resonance of the single scatterer. We believe that these results bear directly on the well-known problem of grating anomalies, which has been discussed analytically by Artman for the case of a reflecting grating. But the results also indicate that anomalies are conceivably not a necessary consequence of resonance in the case of conducting cylinders; the shape of the cylinders may be equally significant.

Before leaving the subject of the above method, which has been discussed here primarily in connection with gratings and slits to which it has been successfully applied, we should like to mention that the method seems applicable to such problems as diffraction by a pair of parallel, long but finite plates and by a finite cylinder.

The grating problem for conducting cylinders of arbitrary cross-section is also approached by making an assumption as to current distribution on the individual elements and by using the variational method to obtain the diffracted field (Shmoys, EM-18). The problem is formulated in terms of characteristic plane waves corresponding to the spectra of various orders defined in optics. Scattering matrix elements are expressed as stationary functionals of current distribution on the grating wires, for the incident wave falling at right angles to the grating elements and polarized either parallel or perpendicular to them. Similar expressions are obtained for the impedance matrix elements; these are evaluated for the thin wire grating, and an equivalent network constructed.

Variational expressions for the scattering matrix of a wire grating obtained in the preceding previous report are evaluated for the case of circularly cylindrical wires (Sollfrey and Shmoys, EM-41). The single scattering approximation is used as a trial field.

The diffraction of a plane wave by an infinite grating of identical circular cylinders has been considered (Reiche, EM-61). It is shown that the various scattered cylindrical waves combine to form a finite number of plane waves propagated in the directions given by the usual grating formula and an infinite number of 'surface waves' propagated along the grating with amplitudes which decrease exponentially with distance from the grating.

In the work on scattering by combinations of obstacles which we did at the outset of our work on the contract we attempted to use the known fields diffracted by single elements such as cylinders and spheres to obtain the total

field by assuming first that each element is excited by the incident field and by the field diffracted just once by other elements. In later papers of this sequence multiple scattering effects were included. In the first of the studies based on this approach (Twersky, EM-22) it was convenient for the purpose of considering certain absorption effects in the scalar or acoustic problem described below to consider the grating as consisting of semi-cylinders and hemispheres on a perfectly conducting or imperfectly conducting plane.

The non-specular reflection of plane waves of sound by various absorbent surfaces composed of either semicylindrical or hemispherical bosses (protuberances) on an infinite plane is analyzed. Approximate solutions in terms of eigenfunctions for the problem of the single boss (with normal impedance Z) on an infinite plane (with normal impedance Z') and a plane wave at an arbitrary angle of incidence are derived through consideration of cylinder or sphere and two simultaneously incident 'image waves.' Finite distributions of small bosses are then treated and the far field solution obtained subject to the restriction that the secondary excitations of the various bosses be neglected. These solutions are found to contain the characteristic Fraunhofer terms for a grating or lattice. The asymptotic solutions for the single bosses ($Kr \gg 1$, $Ka < 1$) are then extended to consider both finite and infinite random distributions. The solutions for the finite distributions are found to contain the characteristic Fraunhofer terms for similarly shaped apertures. The solutions for the infinite distributions (of semicylinders or hemispheres) are found to be remarkably similar when expressed in terms of the volumetric departure from the plane per cm^2 of distribution. The results obtained for the various limiting cases are then compared in the plane of incidence. For certain ranges of the parameters, the results predict the occurrence of a minimum at the specular angle of reflection and the occurrence of some critical angle of incidence for which the reflection at the specular angle is completely specular. The equivalent problems for cylinders and spheres are also considered.

The preceding theory was then extended to the vector or electromagnetic problem and restricted to perfectly conducting bosses on a perfectly conducting plane (Twersky, EM-26). Expressions are obtained also for the ratios of the reflected intensities and radial energy flux polarized parallel and perpendicular to the plane of incidence, as well as for the total intensity and radial energy flux for the case where the incident wave is unpolarized. It is found that for certain values of the parameters the reflected radiation may consist only of either

the specular or the scattered contributions, while for other values of the parameters one of the scattered contributions, either parallel or perpendicular component may vanish. The results also indicate the occurrence of an extremum in the reflected radiation in the vicinity of the specular angle of reflection which for certain ranges of the parameters for the finite distributions of small bosses may be a minimum rather than a maximum. For these cases there is also some critical angle of incidence (not necessarily $\pi/2$ or grazing incidence) for which the reflection at the specular angle is completely specular.

The method of obtaining the total diffracted field of a grating by summing the fields diffracted by each cylinder is extended to include all orders of scattering (Twersky, EM-34). The solution, satisfying any of the usual prescribed boundary conditions at the surface of each cylinder simultaneously, is expressed as the incident wave plus a sum of an infinite number of orders of scattering. Though the general expressions obtained are unwieldy, some results were obtained. It is shown that for a planar configuration, provided the spacing of the cylinders is large compared to wavelength, the multiple scattered contributions are symmetrical with respect to the plane of the cylinders' axes. Hence for the analogous configuration of semicylindrical bosses on a perfectly conducting infinite plane the radiation polarized parallel to the elements is that predicted by single scattering theory, while the perpendicular component (or an acoustic wave and a rigid surface) shows the effects of multiple scattering. An analytic criterion for the range of validity of the single scattering hypothesis is obtained, as well as an approximate solution in closed form which explicitly takes into account all orders of scattering. The wavelengths and values of the parameters for which the effects of multiple scattering are maximal, or indicate the greatest departure from single scattering theory, are determined; the effects being most pronounced when either all orders of scattering are in phase and reinforce to yield an increase of the intensity over that predicted by elementary theory, or when successive orders are completely out of phase and partially annul to yield a decrease.

The general solution obtained in the preceding report is now applied to gratings with spacing large compared to wavelength and radius; also end effects are neglected (Twersky, EM-39). The far field forms of the multiple scattered orders are summed explicitly to obtain solutions in closed form. For radii very small compared to wavelength, approximate solutions are derived and investigated

in detail. The bright and dark bands these indicate (overlaying the usual continuous spectra observed with broadband radiation) are quite similar to the 'grating anomalies' discovered by Wood in 1902. The present analysis provides a simple physical interpretation of their presence in terms of the magnitudes and phases of the various orders of scattering. Simple relations for their dependence on the parameters are derived which are in agreement with experimental results.

The well-known relation between the total energy cross-section of a scatterer and its forward scattered amplitude is extended to obtain an approximate transmission coefficient for a uniform planar distribution of parallel cylinders (Twersky, EM-54). An analogous reflection theorem for an arbitrary cylindrical boss on a perfectly reflecting plane is then derived; here the total cross-section of the boss is related to the scattering amplitude in the specular direction of reflection of the plane. This is extended to obtain an approximate reflection coefficient for a uniform distribution of cylindrical bosses on a plane.

The technique of these preceding few papers was extended to the consideration of random distributions of grating elements (Twersky, EM-58). A formal solution, Ψ, for the wave scattered by a single configuration under a plane incident wave is obtained, and then Ψ and $|\Psi|^2$ are averaged over an ensemble of configurations specified by the probability distribution function for pairs of cylinders. Closed-form approximations for the averages are derived and used to obtain the average energy flux and to discuss certain energy theorems. The results are then applied to the scattered reflection of electromagnetic or acoustic waves from striated surfaces composed of cylindrical bosses on a perfectly reflecting plane. It is shown, for example, that whereas the single-scattered wave may become infinite as grazing incidence is approached, the multiple-scattered wave approaches the negative of the incident wave; thus the results reduce to those for a dielectric plane near grazing incidence. The maximum effects of multiple scattering are discussed, and practical expressions for the reflection coefficients and differential scattering cross sections are obtained.

One paper has been devoted to considering the scattering of a two-dimensional array of grating elements (Twersky, EM-59). It considers the multiple scattering of a plane electromagnetic or acoustic wave by a uniform distribution of cylinders lying parallel to the z-axis in the range $x = 0$ to d, $y = -\infty$ to ∞ (i.e., in an infinite slab). A closed-form representation for the average wave function is derived; then a heuristic procedure is used to obtain the incoherent scattering. Finally certain energy theorems are stated and it

is shown that the results of this paper are in accord with the energy principle.

The function $E(N,\delta) = \sum_{n=1}^{N} n^{-1/2} e^{in\delta}$ arises in diffraction problems involving gratings and in allied antenna problems. δ is the phase difference of neighboring elements and N is the number of elements taken into account. The function E itself is the sum analogue of the Fresnel integral. We write $E = C + iS$ and have calculated (Russek and Twersky, EM-49) curves of S vs C as functions of δ for $N = 1$ to 10 and $N = 10^2, 10^3, 10^4$. It is found that $|E|_{max}$ always occurs for $\delta = 0$, and the curves from $\delta = 0$ to 2π show N-1 loops; for large N, $|E|_{min}$ occurs for $\delta = \pi$. Curves of C and S as functions of δ are also included.

D. Antenna Theory. Since all antenna systems are designed to function with radiation patterns appropriate to the use to which the system is put we have considered the general problem of what radiation patterns are theoretically possible. The first paper treats the scalar problem (Müller, EM-62). A radiation pattern is defined to be the angular factor in the leading term of an asymptotic expansion of a solution of the scalar wave equation; the expansion is in terms of distance from the source and is asymptotic for large distance. The solution itself is also required to satisy a radiation condition. The paper derives a necessary and sufficient condition that a given function be a radiation pattern and proves, in particular, that no (far field) pattern can be identically zero in any sector. The condition also enables one to determine the smallest possible sphere than can contain the sources creating a given pattern.

The corresponding vector problem has also been treated (Müller, EM-95). In this case an electromagnetic radiation pattern is defined as a tangential vector field on the unit sphere; this field describes the asymptotic behavior at infinity of the full electromagnetic field. The square of the modulus of this tangential field determines how the radiated energy depends upon direction. As in the scalar case, a necessary and sufficient condition that a tangential vector field on the sphere be a radiation pattern is derived.

After discussing the properties of radiation patterns the problem of generating these patterns by sources is studied. Each pattern belongs to some electromagnetic field existing outside some sphere and hence it is possibe to generate this pattern by sources contained within the sphere. However these sources are not uniquely determined by the external field and we therefore

consider the question of the most efficient distribution of sources. The concept of efficiency is first defined for a finite array of dipoles located at fixed points. If we assume that these dipoles radiate incoherently the total radiation is easily determined and is, apart from a numerical factor, the sum of the squares of the amplitudes of the dipole fields. However, if the dipoles radiate coherently the fields of the individual dipoles interact and the radiated energy may be made larger or smaller than that obtained in the incoherent case. The efficiency of the distribution of sources is then defined as the ratio of the coherent to the incoherent radiation.

The analysis of efficiency is pursued by using the algebraic theory of quadratic forms. For homogeneous media the coefficients of the forms may be obtained directly. To treat the efficiency of a continuous distribution of dipoles (currents) contained on a line, surface or region, the algebraic quadratic forms are replaced by quadratic integral forms and the theory of linear symmetric integral equations gives a complete description of the efficiency.

It has not been possible to obtain an exact expression for the field collected at the focus or along the caustic surfaces (in the optical sense) of a finite parabolic cylinder or paraboloid. On the other hand, geometrical optics solutions to this problem yield an infinite field at such places though the exact solution is regular there. We have therefore applied our general theory of asymptotic solution to this problem (Kay and Keller, EM-55). An investigation of reflection from cylindrical walls of arbitrary cross sections shows that the occurrence of caustic points means a change in character of the asymptotic expansion of the true field such that the lowest-order term is no longer independent of k but actually contains a factor k raised to a positive power. There also occurs a jump in phase along a ray passing through a caustic which, as is well known, equals $\pi/2$. As an application of the general method of asymptotic solution of electromagnetic problems we consider a plane wave incident on a parabolic cylinder and obtain the lowest order term at the focus and in the neighborhood of the focus. A similar problem for a reflector consisting of a segment of a circular cylinder is also considered in detail.

Another approach to the behavior of fields diffracted by paraboloids and parabolic cylinders, especially in the neighborhood of foci and caustics, uses the special coordinate systems appropriate to these surfaces (Hochstadt, EM-89). In this study too we have sought the high frequency approximation by seeking asymptotic solutions valid for large wave numbers. This technique is applied to

reflections from the interior of parabolic and paraboloidal reflectors. Detailed discussions are then given for the cases of reflection of incoming plane waves, waves emitted by sources at the focus, and waves emitted by sources on the axis of the reflector, for both coordinate systems. For the last problem the nature of the caustic surface and the behavior of the reflected wave on the caustic, off the caustic, and at the cusp of the caustic are discussed. Some properties of the special functions required to work with paraboloidal and parabolic cylinder coordinates such as addition theorems and asymptotic behavior were derived in work done on a related contract.

By making use of the recently expanding knowledge of spheroidal functions we have calculated the field of a quarter-wave dipole antenna above a circular conducting disk of zero thickness and radius a, that is, a finite groundplane (Leitner, EM-19). The current on the groundplane, the radiation resistance, and the radiation pattern of the system for groundplanes of various radii are computed. With these results the distortion of antenna radiation by groundplanes is studied. Good agreement with recent experimental data is found.

4. Wave Propagation

A. Tropospheric Wave Propagation In Smooth Media. Attention was called during World War II to marked refractive and duct effects in the atmosphere on the propagation of ultra-high frequency waves. It was also gradually realized that ducts play a role even at lower frequencies. The normal mode theory used before World War II by van der Pol and Bremmer for homogeneous atmospheres was extended during the war to ultra-high frequency waves in horizontally stratified media and reasonably useful predictions were made possible. However, the extended theory utilized a flat earth and a modified index of refraction which was intended to compensate for the earth-flattening assumption. This theory proved useful in the diffraction zone but calls for the summation of a large number of modes in the lit region and particularly near the horizon. The eigenvalues and eigenfunctions themselves are rather hard to calculate. The continuous spectrum (branch-cut integral) was ignored. In addition, it has been found that the first mode (eigenfunction) alone, which was used to give the field in the diffraction zone, does not suffice to account for the experimental fields obtained during the past few years in the far regions of the diffraction zone. It therefore seemed desirable that the necessarily hasty and approximate work done under the pressure of the war emergency be reexamined and improved.

We have made several studies of the kinds of modes and eigenvalues that can occur in cylindrically and spherically stratified structures. In general these studies determine the complete sets of vector orthogonal functions (eigenfunctions or modes) and their use in the transformation of vector electromagnetic problems into formally solvable ordinary differential equations (Marcuvitz, EM-29 and EM-69). The mode determination is based on a Green's function procedure due to Weyl and others for the explicit evaluation of complete sets of eigenfunctions of differential operators. This formalism was employed to obtain a variety of complete sets of orthogonal functions applicable for alternative representations of electromagnetic fields in spherical regions (EM-29). Each representation has different convergence properties depending upon the parameters involved. Particular applications were made to piecewise constant spherically stratified regions in order to illustrate the phenomenon of surface waves as opposed to the so-called residue waves. The formalism has also been employed to study the mode structure in general cylindrical regions including open and closed regions with variable isotropic and anisotropic media (EM-69).

The mode solution of electromagnetic field problems in such regions requires the knowledge of the appropriate vector eigenfunctions $E_a(x,y)$ and $H_a(x,y)$ distinguishing the cross-sectional or transverse electric and magnetic field behavior of the a-th mode. In general these possess the vector orthogonality property

$$\iint (E_a \cdot H_b \times z_o + H_a \cdot z_o \times E_b)\, dS = \delta_{ab}$$

over the cross-section transverse to the z-axis of the regions. When suitable symmetry obtains this orthogonality property reduces to

$$\iint E_a \cdot H_b \times z_o \, dS = \delta_{ab},$$

and, in conventional cases, this condition reduces to

$$\iint E_a \cdot E_b \, dS = \delta_{ab} \quad \text{or} \quad \iint H_a \cdot H_b \, dS = \delta_{ab}.$$

Generally the complete set of modes involves both a discrete and a continuous spectrum. These papers present the explicit modes E_a, H_a for a number of cases which illustrate the relative quantitative importance of the ordinary and surface

waves of the discrete spectrum (the surface waves are characterized by an exponentially decreasing behavior outside of dielectric regions in which they propagate) and the radiated waves of the continuous spectrum.

A more concrete study of tropospheric propagation over a spherical Earth with a horizontally stratified medium (index a function of distance from the Earth's center) was made to carry the theory to the point where calculations could be undertaken (Friedman, EM-28). The theory is presented so as to be applicable to both tropospheric and ionospheric problems (if the Earth's magnetic field is neglected). The method of transformation of contours used by Watson, Rydbeck and others for special variations is applied to the general case. Some mathematical questions concerning the continuous spectrum, which are usually neglected are examined in detail. The methods of Furry, Pekeris and others for the approximate theory of tropospheric propagation, which assumes a flat earth and a modified index of refraction, is shown to follow from the general theory by making suitable assumptions. The practical problem is reduced to that of finding the eigenvalues of an ordinary differential equation. The W.K.B. method is applied to obtain explicit results for a ductless atmosphere. For the troposphere a new method of obtaining an approximation in the case of a duct is obtained which appears to be suitable for quantitative work.

To obtain a better understanding of the diffracted field which arises presumably by scattering from the Earth's surface in the vicinity of the horizon we considered a special refractive index which, though unrealistic, is mathematically manageable (Friedlander, EM-76).

We treated the propagation of a pulse in a semi-infinite inhomogeneous medium for which the law of variation of the refractive index is such that the wave equation can be solved by the method of separation of variables. The Laplace transform of the solution, giving the steady state field, is first expressed as a series of eigenfunctions, whose asymptotic behavior at distant points of the Laplace-plane is investigated by means of improved variants of the W.K.B. method. An approximation is found whose inverse Laplace transform can be calculated; it applies when a shadow, in the usual sense of geometrical optics, is formed, and it describes the variation of the solution immediately behind the diffracted front, which propagates into the shadow. The work is an extension of the author's recent report on diffraction of pulses by a circular cylinder (EM-64).

Another study of propagation in the lower atmosphere concerns the effect of a sharp change in ground such as occurs when a radio wave passes from land to

sea (Bazer and Karp, EM-46). In particular we consider the field which arises from the diffraction of a plane wave (or ground wave) whose direction of propagation is normal and whose magnetic vector is parallel to the shoreline of the land-sea surface. The sea is treated as a perfect conductor, and on the land the customary impedance boundary condition is imposed. Mathematically, the problem of determining these fields involves the solution of a two-part boundary value problem for the time-reduced wave equation. Exact integral representations of this solution are obtained by means of techniques allied with those of Wiener and Hopf. Far field approximations are found in both the illuminated and shadow regions. Conditions are given which insure the uniqueness of the field. In addition, some comments are made regarding the possibility of coastal refraction in our formulation.

Reference should be made to another study reported under diffraction by dielectric wedges (Karp and Sollfrey, EM-23). There the theory was applied to the wedge formed by a plane oblique to the horizontal ground. However, the change in dielectric constant from one side of the plane to the other simulates a cold-front. Hence the theory of that paper gives results on the passage of a plane wave through a front.

Some attention has been given by electromagnetic research men generally to the important question of whether by choice of a proper 'Earth' and/or by a proper distribution of sources one can concentrate radiation along the Earth at the expense of the field radiated in other directions and thereby increase the effective range of propagation. Application of such theory could be made also in the design of special trapped wave antennas and coated transmission lines (Goubau line). In two papers we have investigated the effect of a dielectric slab over an imperfect Earth, the case of a perfect Earth having been investigated by others outside our group.

The first of these (Lurye, EM-65) analyzes the electromagnetic field produced by a vertical electric dipole located above a dielectric slab, the slab resting on a ground plane of high but finite conductivity. A complex integral representation of the field is first derived. This representation is then used to deduce asymptotic formulas for the various components of the electric and magnetic vectors. The asymptotic formulas are valid for observation points which are more than several wavelengths distant from the dipole. The guided modes associated with the above structure are investigated in detail by means of a certain perturbation technique, with special attention given to the effect of

the finite conductivity of the ground. The connection is established between the lowest order mode in the presence of the slab, and the Sommerfeld surface-wave term associated with the case in which the slab is absent.

The second of these papers (Williams, EM-70) considers the same physical situation with an arbitrary source and obtains explicit expression for the radiation field in terms of the incident field. These expressions are valid when the point of observation is not too near the plane interface. The method is extended to cover the case of one surface deviating slightly from a plane.

The problem of determining the electromagnetic field due to a radiating point dipole at the center of a spherically symmetric medium is formulated in terms of the vector potential (Keller and Keller, EM-16). When the medium is piecewise constant the exact solution is obtained by means of a recursion formula. For the continuously variable medium the problem is reduced to the solution of a Riccati equation. Some special configurations and media are considered and expressions for the reflection and transmission coefficients are obtained.

Among problems which have called for attention is that of scattering and particularly reflection by a rough surface. One idealization suggested itself which seemed simple and yet might be sufficiently general to bear on the practical problem (Magnus, EM-40). A layer of dipoles is used to represent the roughness of a perfectly conducting plane surface. A random distribution of the dipoles is introduced by the assumption that the energy scattered by the dipoles shall show an additive behavior. The parameters of the problem are restricted by the assumption that the distance d between the source of radiation and the point of observation is large. On the other hand, the wavelength may be of the order of magnitude of the roughness, i.e., of the height σ of the scattering dipoles above the plane. Let h and η be the distances from the reflecting plane of the sending dipole and of the point of observation. Then the scattered energy is computed as a function of d, h, η, σ for horizontal and for vertical polarization. The formulas involve an undetermined factor which depends only on the density of the dipole distribution and on the ratio of the amplitudes of the incoming wave and of the wave emitted by a single scattering dipole.

B. Ionospheric Wave Propagation. Though the effect of the ionosphere on electromagnetic wave propagation has been studied for fifty years definitive results are lacking. The reason is that the ionosphere in the presence of the Earth's magnetic field presents a non-homogeneous anisotropic medium with several

parameters such as the gyromagnetic frequency of the electrons, the collision frequency, and density of the electrons, the last two being functions of position, about which quantitative knowledge is fragmentary. Even if the parameters and consequently the dielectric tensor were known precisely the mathematical problem of predicting the exact behavior of a plane wave or dipole field which enters the ionosphere from below is still beyond the full powers of present-day mathematics.

Most papers on the propagation of plane waves in the ionosphere make a number of assumptions at the very outset so as to obtain some simple mathematical forms and therefy some numerical results. However, despite the fact that the exact problem of propagation in a non-isotropic, non-homogeneous stratified medium is quite difficult and that at the present time there is rather little hope of utilizing such theory for numerical results, we did believe it to be worth while to carry the mathematics as far as possible. Utilizing the theory of systems of ordinary differential equations developed as noted above in EM-33, we considered the propagation of plane waves within a stratified ionosphere (H. B. Keller, EM-56).

The ionosphere is assumed to have the distribution $N(z)$ of electrons; and a static Earth's magnetic field, which may vary with z, is assumed to be present. The dielectric property of the ionosphere is then described by a tensor $K(z)$ and Maxwell's equations reduce to a system of four second order partial differential equations in the transverse components E_1, E_2, H_1, and H_2 of the vectors E and H. By assuming that E and H have the forms

$$E = V(z)\, e^{i\frac{\omega}{c}(px+qy)}, \qquad H = I(z)\, e^{i\frac{\omega}{c}(px+qy)}$$

the system of partial differential equations reduces to a system of four first order ordinary differential equations in the components V_1, V_2, I_1, and I_2 of V and I. The system takes the form

$$\frac{dW(z)}{dz} = i\frac{\omega}{c}\, A(z)\, W(z),$$

which is precisely the form studied in EM-33. After a matrix transformation the final form of the solution is then shown to be

$$u_j(z) = \exp\left[\int_{z_0}^{z} r_{jj}(x)dx\right] \cdot \exp\left[\frac{i\omega}{c}\int_{z_0}^{z} \lambda_j(x)dx\right] \cdot C_j(z_0,z), \quad j = 1,2,3,4,$$

where

$$C_j(z_o,z) = \sum_{i=1}^{4} \Omega_{ij}(z)\, u_i(z_o),$$

and where Ω_{ij} is an element of a rather complex matrizant.

The full explanation of the meaning of the terms in this solution is too lengthy to be given here. What is significant about this solution is that each wave consists of three factors; the first is the usual WKB solution for the characteristic waves; the second factor contains the wave behavior; and the third factor contains the continuous interaction or coupling among four waves. Two of these four waves may be identified in special cases as the up-going and down-going ordinary waves and the other two, as the extraordinary waves. In the general case these four waves are coupled with each other continuously. In special cases there are various types of uncoupling. It follows from the theory of this paper that any one of the components of the true E and H in the ionosphere is a linear combination with variable coefficients of the four characteristic waves. Explicit solutions are obtained for normally incident waves with an oblique earth's magnetic field and for oblique incident waves with a vertical earth's magnetic field.

In a note to the preceding paper two points are treated (H.B. Keller, EM-57). In Part I the tensor properties of the ionosphere are derived by a method due to van der Wyck. While we introduce only minor improvements on his method, it was felt worth while to make these results generally more available. Part II shows the equivalence of various forms of the propagation equations. A general scheme is indicated for finding simplifying transformations of the equations in their first-order form. In particular, we obtain that transformation which yields Rydbeck's coupled equations, and give an interpretation.

The relationship between the propagation of electromagnetic waves within the ionosphere to waves incident from below and traveling between the Earth and ionosphere has not been carried out in any generality. However, for special assumptions on the variation of the ionospheric index and for low frequencies it is possible to deduce some results. One of the studies (Shmoys, EM-51) restricted to low frequency waves entering the ionosphere from below and assuming a special form for the ionospheric index takes off from some work by Heading and Whipple. They dealt with the case of a plane wave obliquely incident upon an ionosphere wherein the electron density increases exponentially

with height; the Earth's magnetic field was assumed to be vertical and the collision frequency of ions in the ionosphere assumed to be low. In this case Heading and Whipple solved the system of four equations described above. Our paper treats vertical incidence, oblique magnetic field, and both low and high collision frequency. The results obtained do not agree with experimental data. This is probably due to the fact that coupling effects were neglected. An example is given in the Appendix which shows that the omission of coupling terms, however small their coefficients may be, is not always justified. This means that the assumption of high or low collision frequency will lead to error in the 15-150 kc. range.

For the purpose of determining the effect of the ionosphere on radio reception, the analysis of propagation in the ionosphere itself need not be of direct interest. There is a mathematical approach to scattering which concentrates on the field reflected and transmitted by the scatterer based of course on some knowledge of the scattering medium or obstacle. Hence we have sought to obtain reflection and transmission 'coefficients' for the ionosphere by this method (Lurye, EM-31).

We consider an elliptically polarized plane electromagnetic wave incident obliquely on a continuously stratified anisotropic scattering medium. The medium occupies a region of space between two parallel planes ($0 \leq z \leq a$, say) and is characterized by a magnetic susceptibility which is zero and an electric susceptibility which is a continuous matrix function of z. It is required to obtain the reflected and transmitted plane waves on the incident and far sides of the medium, respectively. The problem is attacked by means of the Schwinger integral equation-variational technique. An integral equation is derived for the electric field vector within the scattering medium. Next, the problem is reduced to computing the elements of a certain 'reflection matrix' and 'transmission matrix.' A set of variational expressions for these matrix elements is then constructed by means of simple considerations in the theory of abstract linear spaces. The variational expressions have the form of functionals depending on two independent vector functions; their Euler equations are the given integral equation and a certain " adjoint" integral equation. Finally, a reciprocity theorem is proved, based on a symmetry property of the variational expressions.

A somewhat simpler approach to the same problem of obtaining reflection and transmission properties of the ionosphere uses the more obvious method of con-

sidering the ionosphere as the limit of a finite number of layers each having a constant index (Russek, EM-38). First, the scattering matrix for an anisotropic multilayer is obtained. This is a possible approximation to the continuously varying medium at low frequencies where each layer is but a fraction of a wavelength in thickness. In the high frequency case, where a solution in series form has already been obtained by Bremmer, it is found that a considerable simplification can be effected by expanding the conductivity matrix in inverse powers of frequency.

Where the problem of the effect of the ionosphere on wave propagation can be simplified by neglecting the Earth's magnetic field, a simplification reasonable at very low frequencies, the mathematics is more manageable. Under this assumption the problem of modes of propagation of electromagnetic waves between a perfectly conducting earth and a gradually varying ionosphere was considered (Shmoys, EM-79). The case of exponentially varying ionospheric parameters, with the electron density starting from 0 near the Earth's surface, is solved by wave theory rather than by ray theory. The propagation constant, the angle and time of arrival, and the group velocity are calculated for the first few modes of propagation. It is shown that the results for the phase and group velocities and angle of arrival of low-order modes obtained when the ionosphere is assumed to be a perfectly conducting sheet at a height simply related to ionospheric parameters are very close to the experimental values.

In a second paper which neglects the effect of the Earth's magnetic field the difficulties with the geometrical optics ('classical') definition of virtual height are discussed (Shmoys, EM-27). The more general definition of virtual height in terms of the derivative of the phase of the reflection coefficient with respect to frequency is derived. It is then shown that the former definition can be derived from the latter, if phase integral method is used. The two definitions are compared in the specific examples of linear, rectangular, Epstein, and parabolic charge distributions. It is demonstrated, by means of examples, that the relation between virtual height and frequency derivative of phase is not valid when the reflected wave contains more than one pulse. In this case the frequency derivative of phase cannot be interpreted as the time delay of any one of the pulses. The time of travel of the pulse is approximately equal to the derivative of the phase of the reflection coefficient with respect to the frequency if a) the magnitude of the reflection coefficient varies slowly with the frequency and b) internal reflections within the ionospheric layer are

neglibible. (The latter assumption has not been generally recognized.) If both assumptions are satisfied and if the electron density varies sufficiently slowly so that the WKB approximation can be used to calculate the reflection coefficient we obtain an expression for the time delay (reflection time) identical with that calculated by others on the basis of group velocity considerations.

A great deal of past and current research on the influence of the ionosphere has utilized the time delay of a signal reflected from the ionosphere to obtain the virtual heights of the ionospheric layers. By obtaining time delay over a range of frequencies it has been possible to infer the variation of the ionospheric index, and to distinguish the effects of the various layers. One of our papers contributes to this phase of work on the ionosphere. The Gauss-Christofel formula is proposed for the numerical evaluation of integrals involving the Chapman distribution (Friedman, EM-17). The method is applied to calculate the virtual height of reflection of vertical incidence pulses as a function of frequency for both the ordinary and extraordinary rays in the cases of transverse (perpendicular to the magnetic field) and longitudinal (along the direction of the magnetic field) propagation. It is also shown that the Chapman distribution may be approximated very closely by a cosine curve. It is indicated how the calculated values of the virtual height may be used to analyze experimental data.

C. The Inverse Propagation Problem. The direct and natural method of calculating the effect of the troposphere or the ionosphere on radio wave propagation is of course to obtain the index of refraction of the entire medium through which the waves may pass and then to tackle the mathematical problem of solving Maxwell's equation in that medium and with a given source. In the case of the ionosphere the impossibility, at least until the recent use of rockets, of obtaining experimental information on the index of that medium has forced the adoption of alternative procedures such as deriving the ionospheric index on the basis of reasonable physical assumptions (the Appleton-Hartree formula) or to the use of time delay information such as was just described. As a consequence of research in the theory of differential equations (see article 2) it occurred to us that we might be able to improve on existing methods of determining the ionospheric index. Our method presupposes a knowledge of the phase of the reflected wave at various frequencies at normal incidence. For non-dispersive media we could alternatively use the reflection

coefficient at single frequency and various angles of incidence. At the present time the method is limited to isotropic media.

We utilize (Kay, EM-74) the theory of the differential equation

$$u''(k,x) + \left[k^2 - V(x)\right] u(k,x) = 0$$

wherein we have shown that if the asymptotic form of the solution at $x = -\infty$ is known, then under various conditions previously described it is possible to determine $V(x)$. This asymptotic form is $e^{ikx} + b(k)e^{-ikx}$ and physically $b(k)$ is the reflection coefficient for the wave e^{ikx} of unit amplitude incident from $-\infty$. The $V(x)$ of the above equation determines the index of refraction of the medium. Hence a knowledge of the reflection coefficient as a function of frequency determines the index. The precise conditions on $b(k)$ and on $V(x)$ which enable one to determine $V(x)$ from a knowledge of $b(k)$ are given. An example, in which $b(k) = -(k^2 + 1)^{-1}$, is worked out in detail. It is also shown how one can obtain $V(x)$ explicitly for a large class of reflection coefficients.

The differential equation $u''(k,x) + k^2n^2(x)\, u(k,x) = 0$ in the interval $-\infty < x < \infty$ is also considered, the problem being to determine the function $n(x)$ from a knowledge of the reflection coefficient.

A more limited investigation in the domain of the inverse propagation problem and applicable to the ionosphere has been completed (Karp and Shmoys, EM-82). A major problem in ionospheric research is that of deducing information on the structure of the ionosphere from soundings by electromagnetic waves. If one neglects collisions and the Earth's magnetic field, the electron density can be calculated from the time of travel of a pulse sent up to the ionosphere and reflected by it, provided this time is known as a function of frequency from zero to some frequency and, what is more important, provided the electron density increases monotonically with height. In the case of multiple layers (those with more than one maximum) the method yields less. We have considered the question: Suppose we had more information than soundings from the ground, how could we utilize them? Continuing to treat propagation in the ionosphere from the group velocity point of view, we could postulate that we have a station above the ionosphere, and that we can make soundings from below, above, and through the ionosphere. The last type of information, the time of transit of a pulse through a valley (a region between maximum ion densities) as a function of frequency

(from penetration frequency to infinity), permits us to calculate the width of the valley at any given electron density. For more complicated electron density distributions we can obtain the cumulative width of the valleys. Since the mathematical procedure involves taking a Mellin transform and later an inverse Mellin transform it is applicable, for the time being, to transit times which are analytic functions of frequency; a suitable approximation procedure would have to be worked out to handle numerical data. Because the above method uses the group velocity only, we cannot obtain the ion distribution uniquely, but the procedure is simpler and yields such information as the depths of valleys, curvature at the bottom, and other quantities.

D. Scattering by Turbulent Air Masses. In the past decade it was found experimentally that radio waves from low to ultra-high frequencies can be propagated over distances many times that previously found and, in the past, attributed to the refractive effect of the ionosphere or the troposphere. The field strengths observed at these greater distances and the large fields observed at smaller distances are now attributed to random scattering by irregularities in the tropospheric and ionospheric indices of refraction. We have recently undertaken to investigate this phenomenon and one paper on this subject reports our results thus far (Silverman, EM-88).

Obukoff's statistical theory of turbulent mixing is proposed as a replacement for the heuristic theories of Gallet and Willars-Weisskopf, and is applied to the problem of the scattering of radio waves by refractive index fluctuations. In the case of ionospheric scattering, order-of-magnitude agreement with the observed scattered power is obtained if the refractive index fluctuations are attributed to electron density fluctuations produced by turbulent mixing in the lower edge of the E-layer. In the case of tropospheric scattering, it appears that order-of-magnitude agreement with the observed scattered power can be obtained, except during the summer months, by attributing the refractive index fluctuations to temperature fluctuations. During the summer months and at low scattering heights, humidity and its fluctuations are expected to play a prominent role. Experimental and theoretical evidence is cited in favor of perennial fractional-degree temperature fluctuations in the troposphere. Comparison of the Obukhoff, Villars-Weisskopf, and Booker-Gordon models is given.

E. Propagation in Wave Guides. Several papers on wave guides resulted from

the realization that our mathematical methods might solve some useful and potentially useful problems. Thus our investigation of the Wiener-Hopf integral equation method coupled with awareness of recent emphasis on wave guides carrying several modes led to the solution of the problem of reflection caused by a change in dimensions of a wave guide, when the larger section supports two modes (Williams, EM-77). We consider the two-dimensional acoustic problem of two channels of infinite width and different heights joined together by a short step (Figure 1), with a wave incident from the right in the larger channel.

It has been found possible to write the reflection coefficient as the sum of two parts, one of which can be written explicitly in terms of known functions, while the other can be obtained as the solution of an infinite number of linear algebraic equations. The explicit part may be obtained by assuming that the step is absent and that the left half of the guide is of the same width as the right but contains a bifurcating plane. We then superimpose onto the incident field an additional field incident from the lower half of the left guide (the shaded region) such that the normal derivative of that part of the total solution involving the fundamental mode vanishes on AB. The incident field to be superimposed may be obtained from a knowledge of the exact solution for the radiation from the lower half of the bifurcated region. The analytic, explicit part of the solution is seen to be by far the dominant part of the complete solution and is such that an approximate solution of the set of linear algebraic equations will yield very good value for the total reflection coefficient. Comparison has been made with the results given by Marcuvitz in the Wave Guide Handbook (McGraw-Hill Publishing Co., N.Y., 1950) and obtained by the static method for the case when the regions are such that only the fundamental mode is incident from the left. The results of both methods agree to within less than 1 %.

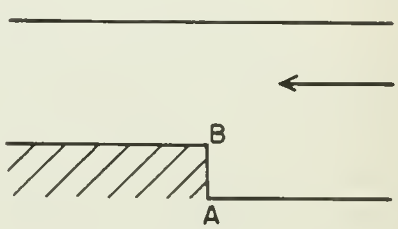

Figure 1

The present work has also been extended to cover the case where the larger section can sustain two non-attentuating modes, and values of the reflection coefficients have been computed for various wave guide and step heights. The method can be extended to cover the case where the step (the shaded region) is replaced by a dielectric slab with a perfectly conducting sheet resting on it;

this is equivalent to a bifurcated guide when one of the bifurcated regions consists of a medium with dielectric properties differing from that of the main structure. The method is also applicable to radiation from dielectric wave guides and it is possible to obtain an iterative type of solution in terms of the difference between the dielectric constants of the dielectric and of free space. In all of these problems it should be possible to recast the solution in such a manner that the dominant part of the solution may be written in terms of known functions whilst the degree of accuracy required for the determination of the other part should be low and yet produce good accuracy for the complete solution.

The application of the reflection principle, discussed in article 2 under equivalence principles, to the theory of wave guides shows how some problems can be solved in terms of others (Karp and Williams, EM-83). Consider the problem of a parallel plate wave guide containing an obstacle (Figure 2) in the form of a thin plate normal to one surface and extending half-way toward the other. We shall call this obstacle a septum. The problem of propagation in such a guide was recently solved by Heins and Baldwin by the Wiener-Hopf technique. However, by means of the reflection principle the solution of this problem can be obtained from the solution of the bifurcated wave guide problem when a plane wave symmetric about the bifurcation is incident (Figure 3).

Figure 2

This same paper shows that a wave guide in the form of a cylinder whose cross-section is uniform and symmetric about an axis and which contains a diaphram (septum) across one half of the cross-section (Figure 4) can be treated if one knows the solution of the problem of the same wave guide but containing a bifurcation along the axis of symmetry D in one half of the infinite guide. The equivalence principle shows also that the first eigenvalue of the problem of propagation

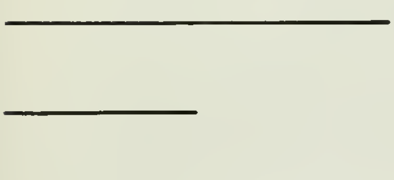

Figure 3

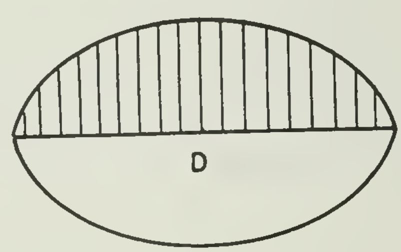

Figure 4

in a wave guide symmetric about two perpendicular axes and containing the septum AB (half of one axis) is the same as the first eigenvalue in the same guide but containing the septum BC (half of the other axis).

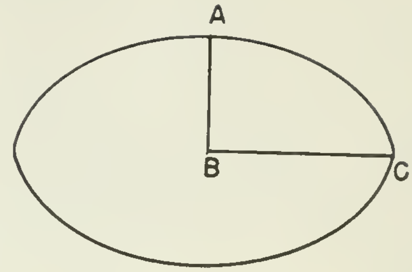

Figure 5

The study of reflectionless transmission through dielectrics, which in itself has been noted to be an application of some theory of differential equations, is equally applicable to the design of reflectionless transmission lines and wave guides, that is, structures which will produce no reflection at all frequencies (Kay and Moses, EM-91).

We should also like to call attention to the papers on the general theory of representation of electromagnetic fields (Marcuvitz, EM-29, 69). These papers are applicable to wave guide structures of various types.

Our attention was drawn to a number of papers by Vajnshtejn in Russian dealing with the application of the Wiener-Hopf method to wave guide problems. It was thought that translation of these papers would be helpful to others as well as to ourselves (Shmoys, EM-63). These papers deal with radiation of acoustic and electromagnetic waves from parallel plane and cylindrical wave guides.

5. Miscellaneous

Four research reports were submitted which do not fall into the above categories. The first of these (Karp, EM-35) is both mathematically and physically significant though falling in the subject of electrostatics. The motivation for this paper came from the study of the scope of the Wiener-Hopf method and since the number of problems that can be solved exactly in electrostatics, as well as in electromagnetics, is limited it was deemed worth while to add this new one to the literature. In this paper the natural charge distribution for a conical cup has been obtained without approximation. Using spherical coordinates the potential u is expressed in the form

$$u = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}\Lambda(\nu)P_{\nu-1}(\cos\theta)d\nu \quad \text{for } 0 < \theta < \theta_0, \quad \text{and}$$

$$u = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu} B(\nu) P_{\nu-1}(-\cos\theta)d\nu \quad \text{for } \theta_o < \theta < \pi.$$

The two-part boundary value problem in r at $\theta = \theta_o$ then gives rise to a single equation between two functions of ν. This equation is solved by Wiener-Hopf techniques applied in the complex $\mu(=\nu-1/2)$ plane. Solution involves: (a) factorization of the product $2P_{-\frac{1}{2}+\mu}(\cos\theta_o) \cdot P_{-\frac{1}{2}+\mu}(\cos\theta_o)/\cos\pi\mu$ in the form $K^+(\mu) \cdot K^-(\mu)$, where $K^+(\mu)$ is regular in a right half-plane and $K^-(\mu)$ in a left half-plane; (b) the asymptotic forms of $K^+(\mu)$, $K^-(\mu)$. The necessary results are obtained by employing: (a) the fact that the zeros of $P_{-\frac{1}{2}+\mu}(\cos\theta_o)$ considered as a function of μ are real, symmetric about the origin and asymptotically in arithmetic progression; (b) comparison of $K_+(\mu)$ with the gamma function of a suitable argument. From the knowledge of the natural charge distribution the capacitance of the conical cup is obtained as well as the behavior of the charge densities at the apex and the circular edge of the cup.

In June 1950 the Air Force Cambridge Research Center and New York University jointly sponsored a Symposium on the Theory of Electromagnetic Waves, which was held at New York University. The papers delivered at this symposium were put together in one volume and submitted as Research Report EM-37. These proceedings were also published by Interscience Publishers of New York under a subvention from New York University. This report and published volume contain among other papers, published versions of EM-21, 24, 25, 28, 29.

The third of these miscellaneous papers is a report on a conference (Friedman, EM-30). At the suggestion of the Office of Naval Research and under the sponsorship of the Air Force Cambridge Research Laboratories the writer attended the conference on 'Dynamics of Ionized Media' held at University College, University of London, March 19-21, 1951. The report is a digest of the proceedings of the conference. This digest is based on handwritten notes taken during the talks and discussion. While considerable effort has been expended to make the report as informative as possible, it was impossible to give a fuller, much less a verbatim, account of the proceedings. It is hoped, nevertheless, that the account will be helpful to coworkers in the field.

6. Complete List of Reports and Publications

The following is a complete list of reports and publications (as of 9/1/56) of work done under Contract No. AF 19(122)-42 by the Division of Electromagnetic Research of the Institute of Mathematical Sciences (formerly Mathematics Research Group, Washington Square College of Arts and Sciences).

Reviews of the reports have been appearing in Mathematical Reviews, and references to these reviews are included.

EM-11	W. Sollfrey	The Variational Solution of Scattering Problems Math. Rev., 11, 482 (1950).	Mar.,1949
EM-12	A. Leitner	Notes on Diffraction by a Circular Disk Published - J. Acoust. Soc. Amer., 21, 331 (1949). Math. Rev., 11, 281 (1950). Math. Rev., 11, 482 (1950).	Apr.,1949
EM-13	J.B. Keller and H.B.Keller	Determination of Reflected and Transmitted Fields by Geometrical Optics Published - J.Opt.Soc.Amer., 40, 48 (1950). Math. Rev., 11, 561 (1950).	May,1949
EM-14	R.K. Luneberg	Asymptotic Development of Steady-State Electromagnetic Fields Math. Rev., 11, 630 (1950).	July,1949
EM-15	R.K. Luneberg	Asymptotic Evaluation of Diffraction Integrals Math. Rev., 12, 305 (1951).	Oct.,1949
EM-16	J.B. Keller and H.B. Keller	A Point Dipole in Spherically Symmetric Media Math. Rev., 12, 224 (1951).	Feb.,1950
EM-17	B. Friedman	Numerical Methods for Evaluation of the Integrals for Virtual Height Math. Rev., 13, 588 (1952).	Feb., 1950
EM-18	J. Shmoys	Diffraction of Electromagnetic Waves by a Plane Wire Grating Published - J. Opt. Soc. Amer., 41, 324 (1951). Math. Rev., 12, 65 (1951).	Mar.,1950

EM-19	A. Leitner	Effect of a Circular Groundplane on Antenna Radiation Published (with R.D. Spence) - J. Appl. Phys., 21, 1001 (1950). Math.Rev., 12, 146 (1951). Math.Rev., 12, 462 (1951).	Apr.,1950
EM-20	J.B. Keller and S. Preiser	Determination of Reflected and Transmitted Fields by Geometrical Optics, Part II Math. Rev., 12, 224 (1951).	Apr.,1950
EM-21	J.B. Keller and A. Blank	Diffraction and Reflection of Pulses by Wedges and Corners Published - Comm.Pure Appl.Math., 4, 75 (1951). (See also EM-37). Math. Rev., 12, 564 (1951).	June,1950
EM-22	V. Twersky	On the Scattered Reflection of Scalar Waves from Absorbent Surfaces Published - J. Acoust.Soc.Amer., 23, 336 (1951) Math.Rev., 12, 650 (1951).	Aug.,1950
EM-23	S. Karp and W. Sollfrey	Diffraction by a Dielectric Wedge with Application to Propagation through a Cold Front Math.Rev., 12, 884 (1951).	Oct.,1950
EM-24	M. Kline	An Asymptotic Solution of Maxwell's Equations Published - Comm.Pure Appl.Math., 4, 225 (1951). (See also EM-37). Math.Rev., 12, 886 (1951).	Nov.,1950
EM-25	S. Karp	Separation of Variables and Wiener-Hopf Techniques Published - Comm.Pure Appl.Math., 3, 411, (1950). (See also EM-37). Math.Rev., 12, 775 (1951). Math.Rev., 13, 134 (1952).	Dec.,1950
EM-26	V. Twersky	On the Scattered Reflection of Electromagnetic Waves Published - J. Appl. Phys., 22, 825 (1951). Math.Rev., 12, 884 (1951).	Jan.,1951

EM-27	J. Shmoys	On the Definition of Virtual Height Published - J.Geophys.Res., 57, 95 (1952). Math.Rev., 12, 885 (1951).	Feb.,1951
EM-28	B. Friedman	The Dipole Field in an Inhomogeneous Atmosphere Published - Comm. Pure Appl. Math., 4, 317 (1951). (See also EM-37). Math.Rev., 13, 305 (1952).	Mar.,1951
EM-29	N. Marcuvitz	Field Representations in Spherically Stratified Regions Published - Comm.Pure Appl.Math., 4, 263 (1951). (See also EM-37).	Apr.,1951
EM-30	B. Friedman	Report on a Conference on Dynamics of Ionized Media Math.Rev., 13, 95 (1952).	May, 1951
EM-31	J. Lurye	Electromagnetic Scattering Matrices of Stratified Anisotropic Media Math.Rev., 13, 606 (1952).	May, 1951
EM-32	W. Magnus	Infinite Matrices Associated with Diffraction by an Aperture Published - Quart.Appl.Math., 11, 77 (1953). Math.Rev., 13, 604 (1952). Math.Rev., 15, 801 (1954).	May, 1951
EM-33	H.B.Keller and J.B. Keller	On Systems of Linear Ordinary Differential Equations Math.Rev., 13, 346 (1952).	July,1951
EM-34	V. Twersky	Multiple Scattering of Radiation, Part I: Arbitrary Configuration of Parallel Cylinders, Planar Configurations, Two Cylinders Sec. 1 published - J.Acoust.Soc.Amer., 24, 42 (1952). Math.Rev., 13, 188 (1952). Sec. 2 and 3 published - J.Appl.Phys., 23, 407 (1952). Math.Rev., 13, 802 (1952).	July,1951

EM-35	S.N. Karp	The Natural Charge Distribution and Capacitance of a Finite Conical Shell Accepted - Quart. Appl. Math. Math.Rev., 13, 802 (1952).	Sept.,1951
EM-36	J.B. Keller	Parallel Reflection of Light by Plane Mirrors Published - Quart.Appl.Math., 11, 216 (1953).	Oct.,1951
EM-37		The Theory of Electromagnetic Waves - A Symposium Published - Interscience Publishers, Inc., N.Y., Dec., 1951. Math.Rev., 13, 707 (1952).	Nov.,1951
EM-38	A. Russek	Scattering Matrices for Ionosphere Models	Dec.,1951
EM-39	V. Twersky	Multiple Scattering of Radiation, Part II (The Grating) Published - J. Appl. Phys., 23, 1099 (1952). Letter-to-Editor - J. Opt. Soc., 42, 855 (1952). Math.Rev., 14, 518 (1953). Math.Rev., 14, 1149 (1953).	Dec.,1951
EM-40	W. Magnus	On the Scattering Effect of a Rough Plane Surface Math.Rev., 14, 933 (1953).	Jan.,1952
EM-41	W. Sollfrey and J. Shmoys	Diffraction of Electromagnetic Waves by a Plane Wire Grating, II Math.Rev., 14, 933 (1953).	Feb.,1952
EM-42	R.S. Phillips	Linear Ordinary Differential Operators of the Second Order Math.Rev., 14, 1088 (1953).	Apr.,1952
EM-43	I. Kay	Diffraction of an Arbitrary Pulse by a Wedge Published - Comm.Pure Appl.Math., 6, 419 (1953). Math.Rev., 14, 1142 (1953) Math.Rev., 15, 321 (1954). Appendix by J.B. Keller - Published - J.Appl. Phys., 23, 1267 (1952).	Apr., 1952
EM-44	B. Friedman and J. Russek	Addition Theorems for Spherical Waves Published - Quart. Appl. Math., 12, 13 (1954). Math.Rev., 14, 1084 (1953). Math.Rev., 15, 702 (1954).	June,1952

EM-45	W. Sollfrey	Diffraction of Pulses by Conducting Wedges and Cones Math.Rev., 14, 1148 (1953).	July,1952
EM-46	J. Bazer and S.N. Karp	Propagation of Plane Electromagnetic Waves Past a Shoreline Math.Rev., 14, 933 (1953).	July,1952
EM-47	B. Friedman	Techniques of Applied Mathematics - Theory of Distributions	Oct., 1952
EM-48	M. Kline	An Asymptotic Solution of Linear Second-order Hyperbolic Differential Equations Published - J.Rational Mechanics, 3, 315 (1954). Math.Rev., 15, 433 (1954) Math.Rev., 15, 800 (1954).	Dec.,1952
EM-49	J. Russek and V. Twersky	Graphs of Function $E(N,\delta) = \sum_{n=1}^{N} n^{-1/2} e^{in\delta}$ Math.Rev., 15, 255 (1954).	Feb.,1953
EM-50	C.J. Bouwkamp	Diffraction Theory - A Critique of Some Recent Developments Math.Rev., 14, 1148 (1953) Math.Rev., 16, 200 (1955). Published in Part, Reports on Progress in Phys., 17, 35 (1954).	Apr.,1953
EM-51	J. Shmoys	Low-Frequency Propagation in an Exponential Ionospheric Layer	Apr., 1953
EM-52	S.N. Karp	Diffraction by a Tipped Wedge - With Applications to Blunt Edges Math.Rev., 15, 375 (1954).	May,1953
EM-53	I. Kay	Diffraction of Pulses by Parabolic Cylinders and Paraboloid of Revolution Math.Rev., 15, 481 (1954).	June,1953
EM-54	V. Twersky	Certain Reflection and Transmission Co-efficients Letter-to-Editor, J.Appl.Phys., 24, 659 (1953) Math.Rev., 15, 844 (1954) Published - J. Appl. Phys., 25, 859 (1954). Math.Rev., 15, 762 (1954) Math.Rev., 16, 201 (1955).	June,1953

EM-55	I. Kay and J.B. Keller	Asymptotic Evaluation of the Field at a Caustic Published - J. Appl. Phys., 25, 876 (1954) Math.Rev., 16, 199 (1955) Math.Rev., 16,544 (1955).	Aug.,1953
EM-56	H.B. Keller	Ionospheric Propagation of Plane Waves Math.Rev., 15, 585 (1954).	Aug.,1953
EM-57	H.B. Keller	On the Electromagnetic Field Equations in the Ionosphere Math.Rev., 15, 585 (1954).	Sept.,1953
EM-58	V. Twersky	Multiple Scattering of Waves by Planar Random Distributions of Parallel Cylinders and Bosses Math.Rev., 15, 585 (1954).	Oct.,1953
EM-59	V. Twersky	Multiple Scattering of Waves by a Volume Distribution of Parallel Cylinders Math.Rev., 15, 762, (1954).	Oct.,1953
EM-60	B. Friedman	Techniques of Applied Mathematics - Ordinary Differential Equations and Green's Functions	Nov.,1953
EM-61	F. Reiche	On Diffraction by an Infinite Grating Math.Rev., 16, 97 (1955).	Dec.,1953
EM-62	C. Müller	Radiation Patterns and Radiation Fields Published - J.Rational Mech., 4, 235 (1955). Math.Rev., 16, 428 (1955).	Mar.,1954
EM-63	J. Shmoys	Propagation in Semi-Infinite Waveguides Translations of Six Papers by L.A.Vajnshtejn Published - Abstract, J.Appl.Phys., 26, 1284 (1955).	Jan.,1954
EM-64	F.G.Friedlander	Diffraction of Pulses by a Circular Cylinder Published - Comm. Pure Appl. Math., 7, 705 (1954) Math.Rev., 16, 87 (1955) Math.Rev., 16, 538 (1955).	Apr.,1954
EM-65	J. Lurye	The Electromagnetic Field of a Dipole over a Dielectric Slab on a Finitely Conducting Ground Plane Math.Rev., 16, 201 (1955).	July,1954

EM-66	J.Bazer and S.N. Karp	Potential Flow through a Conical Pipe with an Application to Diffraction Theory Published - J. Rational Mech., 5, 277 (1956). Math.Rev., 16, 412 (1955).	Aug.,1954
EM-67	F.G.Friedlander and J.B.Keller	Asymptotic Expansions of Solutions of $(\nabla^2+k^2)u = 0$ Published - Comm.Pure Appl.Math., 8, 387 (1955) Math.Rev., 16, 482 (1955) Math.Rev., 17, 41 (1956).	Sept.,1954
EM-68	J. Meixner	Diffraction of Electromagnetic Waves by a Slit in a Conducting Plane Between Different Media Math.Rev., 16, 773 (1955).	Oct.,1954
EM-69	N. Marcuvitz	Field Representations in General Cylindrical Regions. I Math.Rev., 16, 885 (1955).	Nov.,1954
EM-70	W.E. Williams	Reflection and Refraction of Electromagnetic Waves by a Dielectric Slab between Dielectric Media Accepted - J. Math. and Phys.	Nov.,1954
EM-71	S.N. Karp	The Effect of Discontinuities of Dielectric Constant on Electrostatic Fields near Conductors Math.Rev., 16, 885 (1955).	Dec.,1954
EM-72	J. Meixner	The Behavior of Electromagnetic Fields at Edges Math.Rev., 16, 885 (1955).	Dec.,1954
EM-73	N. Chako	On Integral Relations Involving Products of Spheroidal Functions Accepted - J. Math. and Phys. Math. Rev., 16, 1107 (1955).	Jan., 1955
EM-74	I. Kay	The Inverse Scattering Problem Math. Rev., 16, 1113 (1955).	Feb., 1955
EM-75	S.N. Karp and A. Russek	Diffraction by a Wide Slit Published - J. Appl. Phys., 27, 886 (1956). Math.Rev., 17, 433 (1956).	Feb.,1955

EM-76	F.G.Friedlander	Propagation of a Pulse in an Inhomogeneous Medium Math. Rev., <u>16</u>, 977 (1955).	Mar.,1955
EM-77	W.E. Williams	Step Discontinuities in Waveguides <u>Submitted</u> - Trans. of Prof. Group on Antennas and Prop. of IRE. Math.Rev., <u>16</u>, 1181 (1955).	Apr.,1955
EM-78	D.S. Jones	A Critique of the Variational Method in Scattering Problems <u>Accepted</u> - Trans. of Prof. Group on Antennas and Prop. of IRE. Math.Rev., <u>16</u>, 1175 (1955).	May,1955
EM-79	J. Shmoys	Long-Range Propagation of Low-Frequency Radio Waves Between the Earth and the Ionosphere <u>Published</u> - Proc. I.R.E., <u>44</u>, 163 (1956). Math.Rev., <u>17</u>, 109 (1956).	May,1955
EM-80	W. Magnus	An Infinite System of Linear Equations Arising in Diffraction Theory Math.Rev., <u>17</u>, 165 (1956).	June,1955
EM-81	J.B. Keller, R.M. Lewis and B.D. Seckler	Asymptotic Solution of Some Diffraction Problems <u>Published</u> - Comm.Pure Appl.Math., <u>9</u>, 207 (1956). Math.Rev., <u>17</u>, 41 (1956).	June,1955
EM-82	S.N. Karp and J. Shmoys	Calculation of Charge Density Distribution of Multilayers from Transit Time Data <u>Published</u> - J. Geophys. Res., <u>61</u>, 183 (1956). Appendix submitted - J. Appl. Phys.	July,1955
EM-83	S.N. Karp and W.E. Williams	Equivalence Relations in Diffraction Theory <u>Accepted</u> - Proc. Camb. Philos. Soc.	Sept.,1955
EM-84	H. Levine	Diffraction by a Circular Aperture at High Frequencies <u>Accepted</u> - Comm.Pure Appl.Math.	Sept.,1955
EM-85	S.N. Karp	Diffraction by an Infinite Grating of Arbitrary Cylinders	Oct.,1955
EM-86	G. Kear	The Forward Scattering of High-Frequency Plane Waves by a Sphere	Nov.,1955

EM-87	D.S. Jones	A New Method for Calculating Scattering with Particular Reference to the Circular Disc Accepted - Comm.Pure Appl.Math.	Dec.,1955
EM-88	R.A. Silverman	Turbulent Mixing Theory Applied to Radio Scattering Published - J. Appl. Phys., 27, 699 (1956).	Jan.,1956
EM-89	H. Hochstadt	Asymptotic Formulas for the Diffraction by Parabolic Surfaces	Mar.,1956
EM-90	S.N. Karp and J. Radlow	On Resonance in Infinite Gratings of Cylinders Accepted - Trans. of Prof. Group on Antennas and Propagation of I.R.E.	Apr.,1956
EM-91	I. Kay and H.E. Moses	Reflectionless Transmission Through Dielectrics and Scattering Potentials Accepted - J. Appl. Phys.	May,1956
EM-92	J.B. Keller	Diffraction by an Aperture. I	June,1956
EM-93	M. Kline	Electromagnetic Research at the Institute of Mathematical Sciences of New York University Published - I.R.E. Transactions on Antennas and Propagation, Symposium on Electromagnetic Wave Theory, AP-4, 243 (1956).	July,1956
EM-94	J.B. Keller	Diffraction by a Convex Cylinder Published - I.R.E. Transactions on Antennas and Propagation, Symposium on Electromagnetic Wave Theory, AP-4, 312 (1956).	July,1956
EM-95	C. Müller	Electromagnetic Radiation Patterns and Sources Published - I.R.E. Transactions on Antennas and Propagation, Symposium on Electromagnetic Wave Theory, AP-4, 224 (1956).	July,1956
EM-96	J.B. Keller, R.M. Lewis and B.D. Seckler	Diffraction by an Aperture.II	Aug.,1956

<u>List of Publications not based on Reports</u>

Lester Kraus	Letter-to-Editor, J. Geophys. Res., <u>58</u>, 551 (1953).
K.O. Friedrichs and J.B. Keller	Geometrical Acoustics.II. Diffraction,Reflection, and Refraction of a Weak Spherical or Cylindrical Shock at a Plane Interface, J. of Appl. Phys., <u>26</u>, 961 (1955).

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Basic research in electromagnetic theory.

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magnetic theory.